Euler-Bernoulli Field-Consistent Beam(Frame) Using Total Lagrangian Formulation

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1　はじめに

有限回転のような幾何学的非線形な梁を扱うために、Euler-Bernoulli 梁の Total Lagrange 法による定式化を行う。
Euler-Bernoulli 梁の仮定の下では、X,Y の変位と Z まわりの回転角は独立ではない。
変位と回転角の関係式を満たすには通常の形状関数を用いて補間することはできず、field-consistent な補間関数を構成しなければならない。
field-consistent な補間関数は、節点値に関して線形にはならない。

2　Kinematics

初期位置 C_0 は X 軸に沿っている場合について定式化する（ローカルな定式化)。
変形によって初期位置 C_0 の点 $P_0(X, Y)$ が現在位置 C の点 $P(x, y)$ に移動するものとする。
Euler-Bernoulli の仮定の下では、

$$x = X + u_X - Y \sin\theta \tag{1}$$
$$y = u_Y + Y \cos\theta \tag{2}$$

u_X、u_Y は $P_0(X, Y)$ を中立軸 (neutral axis) へ投影した点の変位であり、θ は断面の回転角である。
現在位置における中立軸の微小長さを $\mathrm{d}s$ とすると、

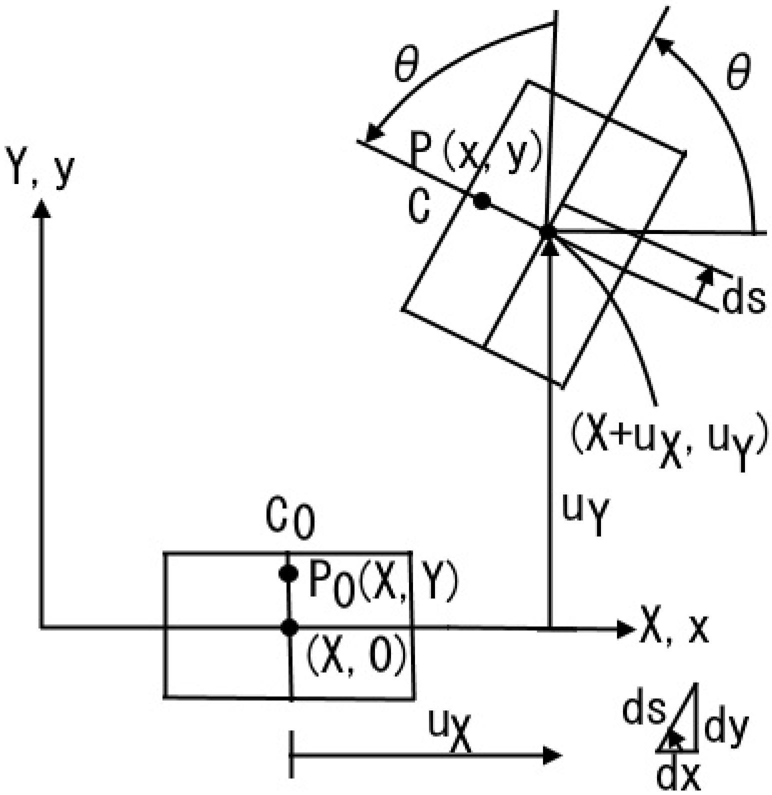

図 1　Kineatics

$$
\begin{aligned}
\mathrm{d}x &= \mathrm{d}s\cos\theta \\
\mathrm{d}y &= \mathrm{d}s\sin\theta
\end{aligned}
$$

$$\rightarrow$$

$$
\begin{aligned}
\frac{\partial x}{\partial X} &= \frac{\partial s}{\partial X}\cos\theta \\
\frac{\partial y}{\partial X} &= \frac{\partial s}{\partial X}\sin\theta
\end{aligned}
$$

中立軸上では、

$$
\begin{aligned}
x &= X + u_X \\
y &= u_Y
\end{aligned}
$$

であるから、

$$
\begin{aligned}
1 + u'_X &= s'\cos\theta \\
u'_Y &= s'\sin\theta && (3)\\
s' &= \sqrt{(1+u'_X)^2 + (u'_Y)^2} && (4)\\
\cos\theta &= \frac{1+u'_X}{\sqrt{(1+u'_X)^2+(u'_Y)^2}} \\
\sin\theta &= \frac{u'_Y}{\sqrt{(1+u'_X)^2+(u'_Y)^2}} \\
\tan\theta &= \frac{u'_Y}{1+u'_X} && (5)
\end{aligned}
$$

ここに、prime は X についての偏微分である。

変形勾配テンソル $\boldsymbol{F}$ を求める。

$$
\boldsymbol{F} = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{bmatrix} = \begin{bmatrix} 1 + u'_X - Y\theta'\cos\theta & -\sin\theta \\ u'_Y - Y\theta'\sin\theta & \cos\theta \end{bmatrix} \tag{6}
$$

Green-Lagrange ひずみ $\boldsymbol{E}$ を求める。

$$
\boldsymbol{E} = \begin{bmatrix} E_{XX} & E_{XY} \\ E_{YX} & E_{YY} \end{bmatrix} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}) \tag{7}
$$

ここで、

$$
\begin{aligned}
\boldsymbol{F}^T\boldsymbol{F} &= \begin{bmatrix} F_{XX} & F_{YX} \\ F_{XY} & F_{YY} \end{bmatrix}\begin{bmatrix} F_{XX} & F_{XY} \\ F_{YX} & F_{YY} \end{bmatrix} \\
&= \begin{bmatrix} F_{XX}^2 + F_{YX}^2 & F_{XX}F_{XY} + F_{YX}F_{YY} \\ F_{XY}F_{XX} + F_{YY}F_{YX} & F_{XY}^2 + F_{YY}^2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
F_{XX}^2 + F_{YX}^2 &= (1 + u'_X - Y\theta'\cos\theta)^2 + (u'_Y - Y\theta'\sin\theta)^2 \\
&= 1 + 2(u'_X - Y\theta'\cos\theta) + (u'_X - Y\theta'\cos\theta)^2 + (u'_Y - Y\theta'\sin\theta)^2
\end{aligned}
$$

$$
\begin{aligned}
F_{XX}F_{XY} + F_{YX}F_{YY} &= (1 + u'_X - Y\theta'\cos\theta)(-\sin\theta) + (u'_Y - Y\theta'\sin\theta)\cos\theta \\
&= -(1 + u'_X)\sin\theta + u'_Y\cos\theta \\
F_{XY}^2 + F_{YY}^2 &= (-\sin\theta)^2 + (\cos\theta)^2 = 1
\end{aligned}
$$

ここで、

$$
-(1 + u'_X)\sin\theta + u'_Y\cos\theta = 0 \tag{8}
$$

の関係 ((5) 式から成立することが確かめられる) があるから、

$$
\boldsymbol{E} = \frac{1}{2}\begin{bmatrix} 2(u'_X - Y\theta'\cos\theta) + (u'_X - Y\theta'\cos\theta)^2 + (u'_Y - Y\theta'\sin\theta)^2 & 0 \\ 0 & 0 \end{bmatrix} \tag{9}
$$

$\boldsymbol{E}$ の線形化を行う。
直交行列 $\boldsymbol{R}$ を導入する。

$$
\boldsymbol{R}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{10}
$$

$$
\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}
$$

$$
\boldsymbol{R}(-\theta) = \boldsymbol{R}(\theta)^T
$$

$\boldsymbol{R}$ は Z 軸周りの回転を表現する。

$$
\begin{aligned}
\boldsymbol{F} &= \begin{bmatrix} 1 + u'_X - Y\theta'\cos\theta & -\sin\theta \\ u'_Y - Y\theta'\sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} 1 + u'_X & 0 \\ u'_Y & 0 \end{bmatrix} + \boldsymbol{R}(-\theta)\begin{bmatrix} -Y\theta' & 0 \\ 0 & 1 \end{bmatrix} \qquad (11)
\end{aligned}
$$

新たに $\overline{\boldsymbol{F}}$ を定義する。

$$
\overline{\boldsymbol{F}} = \boldsymbol{R}\boldsymbol{F} \tag{12}
$$

$$
\begin{aligned}
&= \boldsymbol{R}\begin{bmatrix} 1 + u'_X & 0 \\ u'_Y & 0 \end{bmatrix} + \boldsymbol{R}\boldsymbol{R}(-\theta)\begin{bmatrix} -Y\theta' & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} (1 + u'_X)\cos\theta + u'_Y\sin\theta & 0 \\ -(1 + u'_X)\sin\theta + u'_Y\cos\theta & 0 \end{bmatrix} + \begin{bmatrix} -Y\theta' & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} (1 + u'_X)\cos\theta + u'_Y\sin\theta - Y\theta' & 0 \\ 0 & 1 \end{bmatrix} \qquad (13)
\end{aligned}
$$

((8) 式を用いた)

$$
= \begin{bmatrix} s'\cos^2\theta + s'\sin^2\theta - Y\theta' & 0 \\ 0 & 1 \end{bmatrix}
$$

((3) 式を用いた)

$$
\begin{aligned}
&= \begin{bmatrix} s' - Y\theta' & 0 \\ 0 & 1 \end{bmatrix} \\
&= \boldsymbol{L} + \boldsymbol{I} \qquad (14)
\end{aligned}
$$

$$
\boldsymbol{L} = \begin{bmatrix} s' - 1 - Y\theta' & 0 \\ 0 & 0 \end{bmatrix} \tag{15}
$$

$\boldsymbol{L}$ はこれから示すように線形項になる。
$\boldsymbol{R}$ が直交行列であるから、

$$\boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{F}^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{F} = \overline{\boldsymbol{F}}^T\overline{\boldsymbol{F}}$$

となる。これを使うと、

$$\begin{aligned}
\boldsymbol{E} &= \frac{1}{2}(\overline{\boldsymbol{F}}^T\overline{\boldsymbol{F}} - \boldsymbol{I}) \\
&= \frac{1}{2}\left\{(\boldsymbol{L}^T + \boldsymbol{I})(\boldsymbol{L} + \boldsymbol{I}) - \boldsymbol{I}\right\} \\
&= \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T + \boldsymbol{L}^T\boldsymbol{L}) \\
&\simeq \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T) = \boldsymbol{L} \qquad (16)
\end{aligned}$$

小さなひずみを仮定すると $\boldsymbol{L}$ の $(s'-1)$ と $Y\theta'$ が小さいので、$\boldsymbol{L}^T\boldsymbol{L}$ の項は無視できるとした。
E_{XX} が唯一の非 0 ひずみ成分である。

$$\begin{aligned}
E_{XX} &= (1 + u'_X)\cos\theta + u'_Y\sin\theta - Y\theta' - 1 \qquad (17) \\
&= s' - 1 - Y\theta' \qquad (18)
\end{aligned}$$

3 Total Lagrange 法

Total Lagrange 法における仮想仕事式は、

$$\int_V S_{XX}\delta E_{XX}dV = \sum_p F_{Xp}\delta u_{Xp} + \sum_p F_{Yp}\delta u_{Yp} + \sum_p M_{Zp}\delta\theta_{Zp} \qquad (19)$$

左辺

$$\begin{aligned}
\int_V S_{XX}\delta E_{XX}dV &= \int_L\int_A S_{XX}\delta E_{XX}dAdX \\
&= \int_L\int_A EE_{XX}\delta E_{XX}dAdX \qquad (20)
\end{aligned}$$

S_{XX} は第 2Piola-Kirchhoff 応力テンソル $\boldsymbol{S}$ の XX 成分であり、

$$S_{XX} = EE_{XX} \qquad (21)$$

（E:Young 率）の関係がある。
F_{Xp}, F_{Yp}, M_{Zp} $(p = 1, 2)$ は梁の 2 つの端における外力である。
E_{XX} の第 1 変分を求める。

$$\begin{aligned}
\delta E_{XX} &= \delta\left[(1 + u'_X)\cos\theta + u'_Y\sin\theta - Y\theta' - 1\right] \\
&= \cos\theta\delta u'_X - (1 + u'_X)\sin\theta\delta\theta + \sin\theta\delta u'_Y + u'_Y\cos\theta\delta\theta - Y\delta\theta' \\
&= \cos\theta\delta u'_X + \sin\theta\delta u'_Y + \left\{-(1 + u'_X)\sin\theta + u'_Y\cos\theta\right\}\delta\theta - Y\delta\theta' \\
&= \cos\theta\delta u'_X + \sin\theta\delta u'_Y - Y\delta\theta' \qquad (22)
\end{aligned}$$

((8) 式を用いた)

$$\begin{aligned}
E_{XX}\delta E_{XX} &= (s' - 1 - Y\theta')(\cos\theta\delta u'_X + \sin\theta\delta u'_Y - Y\delta\theta') \\
&= s'\cos\theta\delta u'_X + s'\sin\theta\delta u'_Y - s'Y\theta' \\
&\quad - \cos\theta\delta u'_X - \sin\theta\delta u'_Y + Y\delta\theta' \\
&\quad -Y\theta'\cos\theta\delta u'_X - Y\theta'\sin\theta\delta u'_Y + Y^2\theta'\delta\theta' \qquad (23)
\end{aligned}$$

ここで、

$$\int_A dA = A,\ \int_A Y dA = 0,\ \int_A Y^2 dA = I \qquad (24)$$

であるから、

$$\begin{aligned}
\int_V S_{XX}\delta E_{XX} dV &= EA\int_L s'\cos\theta\delta u'_X dX + EA\int_L s'\sin\theta\delta u'_Y dX \\
&\quad -EA\int_L \cos\theta\delta u'_X dX - EA\int_L \sin\theta\delta u'_Y dX \\
&\quad +EI\int_L \theta'\delta\theta' dX \\
&= EA\int_L (1+u'_X)\delta u'_X dX + EA\int_L u'_Y\delta u'_Y dX \\
&\quad -EA\int_L \cos\theta\delta u'_X dX - EA\int_L \sin\theta\delta u'_Y dX \\
&\quad +EI\int_L \theta'\delta\theta' dX \qquad (25)
\end{aligned}$$

(5) 式の

$$\tan\theta = \frac{u'_Y}{1+u'_X}$$

を微分すると、

$$\begin{aligned}
\frac{1}{\cos^2\theta}\theta' &= -\frac{u''_X}{(1+u'_X)^2}u'_Y + \frac{u''_Y}{1+u'_X} \\
&= \frac{1}{(1+u'_X)^2}\{(1+u'_X)u''_Y - u'_Y u''_X\} \qquad (26)
\end{aligned}$$

よって、

$$\begin{aligned}
\theta' &= \frac{\cos^2\theta}{(1+u'_X)^2}\{(1+u'_X)u''_Y - u'_Y u''_X\} \\
&= \frac{(1+u'_X)u''_Y - u'_Y u''_X}{(1+u'_X)^2 + (u'_Y)^2} \qquad (27)
\end{aligned}$$

これを用いると、

$$
\begin{aligned}
\int_V S_{XX}\delta E_{XX}dV = EA\int_L (1+u'_X)\delta u'_X dX + EA\int_L u'_Y \delta u'_Y dX \\
-EA\int_L \frac{1+u'_X}{\sqrt{(1+u'_X)^2+(u'_Y)^2}}\delta u'_X dX - EA\int_L \frac{u'_Y}{\sqrt{(1+u'_X)^2+(u'_Y)^2}}\delta u'_Y dX \\
+EI\int_L \frac{(1+u'_X)u''_Y - u'_Y u''_X}{(1+u'_X)^2+(u'_Y)^2}\delta\theta' dX
\end{aligned} \tag{28}
$$

梁要素の端点を 1、2 とし、中点を 3 としたとき、節点変位ベクトル $\boldsymbol{U}$ と節点力ベクトル $\boldsymbol{f}$ を次のように定義する。

$$
\boldsymbol{U} = \begin{bmatrix} u_{X1} & u_{Y1} & \theta_{Z1} & u_{X2} & u_{Y2} & \theta_{Z2} & u_{X3} \end{bmatrix}^T \tag{29}
$$

$$
\boldsymbol{f} = \begin{bmatrix} F_{X1} & F_{Y1} & M_{Z1} & F_{X2} & F_{Y2} & M_{Z2} & 0 \end{bmatrix}^T \tag{30}
$$

また、形状関数に対応する量を次のように定義する。

$$
\boldsymbol{N}_X = \begin{bmatrix} \frac{\partial u_X}{\partial u_{X1}} & \frac{\partial u_X}{\partial u_{Y1}} & \frac{\partial u_X}{\partial \theta_{Z1}} & \frac{\partial u_X}{\partial u_{X2}} & \frac{\partial u_X}{\partial u_{Y2}} & \frac{\partial u_X}{\partial \theta_{Z2}} & \frac{\partial u_X}{\partial u_{X3}} \end{bmatrix}^T \tag{31}
$$

$$
\boldsymbol{N}_Y = \begin{bmatrix} \frac{\partial u_Y}{\partial u_{X1}} & \frac{\partial u_Y}{\partial u_{Y1}} & \frac{\partial u_Y}{\partial \theta_{Z1}} & \frac{\partial u_Y}{\partial u_{X2}} & \frac{\partial u_Y}{\partial u_{Y2}} & \frac{\partial u_Y}{\partial \theta_{Z2}} & \frac{\partial u_Y}{\partial u_{X3}} \end{bmatrix}^T \tag{32}
$$

これらを用いると、u_X、u_Y の第 1 変分は、

$$
\begin{aligned}
\delta u_X &= \boldsymbol{N}_X^T \delta\boldsymbol{U} \\
\delta u_Y &= \boldsymbol{N}_Y^T \delta\boldsymbol{U}
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
\delta u'_X &= \boldsymbol{N}'_X{}^T \delta\boldsymbol{U} \\
\delta u'_Y &= \boldsymbol{N}'_Y{}^T \delta\boldsymbol{U}
\end{aligned} \tag{34}
$$

となる。

θ の第 1 変分は、(5) 式の

$$
\tan\theta = \frac{u'_Y}{1+u'_X}
$$

より、

$$
\begin{aligned}
\frac{1}{\cos^2\theta}\delta\theta &= -\frac{\delta u'_X}{(1+u'_X)^2}u'_Y + \frac{\delta u'_Y}{1+u'_X} \\
\delta\theta &= \frac{\cos^2\theta}{(1+u'_X)^2}\{(1+u'_X)\delta u'_Y - u'_Y\delta u'_X\} \\
&= -\frac{u'_Y}{(1+u'_X)^2+(u'_Y)^2}\delta u'_X + \frac{1+u'_X}{(1+u'_X)^2+(u'_Y)^2}\delta u'_Y
\end{aligned} \tag{35}
$$

(34) 式を用いると、

$$\delta\theta = \left(-\frac{u'_Y}{(1+u'_X)^2+(u'_Y)^2}\boldsymbol{N}'_X{}^T + \frac{1+u'_X}{(1+u'_X)^2+(u'_Y)^2}\boldsymbol{N}'_Y{}^T \right)\delta\boldsymbol{U} \tag{36}$$

$$\delta\theta = \boldsymbol{N}_\theta{}^T\delta\boldsymbol{U} \tag{37}$$

$$\boldsymbol{N}_\theta = -\frac{u'_Y}{(1+u'_X)^2+(u'_Y)^2}\boldsymbol{N}'_X + \frac{1+u'_X}{(1+u'_X)^2+(u'_Y)^2}\boldsymbol{N}'_Y \tag{38}$$

$$\delta\theta' = \boldsymbol{N}'_\theta{}^T\delta\boldsymbol{U} \tag{39}$$

これら第 1 変分の X 微分を (28) 式に代入すると、

$$\begin{aligned}\int_V S_{XX}\delta E_{XX}dV &= EA\int_L (1+u'_X)\boldsymbol{N}'_X{}^T dX\delta\boldsymbol{U} + EA\int_L u'_Y\boldsymbol{N}'_Y{}^T dX\delta\boldsymbol{U}\\ &\quad -EA\int_L \frac{1+u'_X}{\sqrt{(1+u_X)^2+(u'_Y)^2}}\boldsymbol{N}'_X{}^T dX\delta\boldsymbol{U} - EA\int_L \frac{u'_Y}{\sqrt{(1+u'_X)^2+(u'_Y)^2}}\boldsymbol{N}'_Y{}^T dX\delta\boldsymbol{U}\\ &\quad +EI\int_L \frac{(1+u'_X)u''_Y - u'_Y u''_X}{(1+u'_X)^2+(u'_Y)^2}\boldsymbol{N}'_\theta{}^T dX\delta\boldsymbol{U} \qquad (40)\\ &= \delta\boldsymbol{U}^T\boldsymbol{\Phi}(\boldsymbol{U}) \qquad (41)\end{aligned}$$

梁の剛性方程式 (stiffness equation) は、

$$\boldsymbol{\Phi}(\boldsymbol{U}) = \boldsymbol{f} \tag{42}$$

ここに、

$$\begin{aligned}\boldsymbol{\Phi}(\boldsymbol{U}) &= EA\int_L (1+u'_X)\boldsymbol{N}'_X dX + EA\int_L u'_Y\boldsymbol{N}'_Y dX\\ &\quad -EA\int_L \frac{1+u'_X}{\sqrt{(1+u'_X)^2+(u'_Y)^2}}\boldsymbol{N}'_X dX - EA\int_L \frac{u'_Y}{\sqrt{(1+u'_X)^2+(u'_Y)^2}}\boldsymbol{N}'_Y dX\\ &\quad +EI\int_L \frac{(1+u'_X)u''_Y - u'_Y u''_X}{(1+u'_X)^2+(u'_Y{}^2)}\boldsymbol{N}'_\theta dX \qquad (43)\end{aligned}$$

(42) 式は非線形方程式であり、Newton-Raphson 法で解く。

4 接線剛性行列

Newton-Raphson 法を用いると、(42) 式

$$\boldsymbol{\Phi}(\boldsymbol{U}) = \boldsymbol{f}$$

は次のように線形化できる。

$$\boldsymbol{K}\Delta\boldsymbol{U} = \boldsymbol{f} - \boldsymbol{\Phi} \tag{44}$$

ここにローカル接線剛性行列 $\boldsymbol{K}$ は、

$$\boldsymbol{K} = \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{U}} \tag{45}$$

$$\begin{aligned}
K_{ij} &= \frac{\partial \Phi_i}{\partial U_j} \\
&= EA \int_L \frac{\partial c_1}{\partial U_j} N'_{Xi} + c_1 \frac{\partial N'_{Xi}}{\partial U_j} dX \\
&\quad + EA \int_L \frac{\partial c_2}{\partial U_j} N'_{Yi} + c_2 \frac{\partial N'_{Yi}}{\partial U_j} dX \\
&\quad - EA \int_L \frac{\partial c_3}{\partial U_j} N'_{Xi} + c_3 \frac{\partial N'_{Xi}}{\partial U_j} dX \\
&\quad - EA \int_L \frac{\partial c_4}{\partial U_j} N'_{Yi} + c_4 \frac{\partial N'_{Yi}}{\partial U_j} dX \\
&\quad + EI \int_L \frac{\partial c_5}{\partial U_j} N'_{\theta i} + c_5 \frac{\partial N'_{\theta i}}{\partial U_j} dX
\end{aligned} \tag{46}$$

$$c_1 = 1 + u'_X \tag{47}$$

$$c_2 = u'_Y \tag{48}$$

$$c_3 = \frac{1 + u'_X}{\sqrt{(1 + u'_X)^2 + (u'_Y)^2}} \tag{49}$$

$$c_4 = \frac{u'_Y}{\sqrt{(1 + u'_X)^2 + (u'_Y)^2}} \tag{50}$$

$$c_5 = \frac{(1 + u'_X) u''_Y - u'_Y u''_X}{(1 + u'_X)^2 + (u'_Y)^2} \tag{51}$$

$$\frac{\partial c_1}{\partial U_j} = \frac{\partial u'_X}{\partial U_j} \tag{52}$$

$$\frac{\partial c_2}{\partial U_j} = \frac{\partial u'_Y}{\partial U_j} \tag{53}$$

$$\frac{\partial c_3}{\partial U_j} = \frac{-(1 + u'_X) u'_Y \dfrac{\partial u'_Y}{\partial U_j} + (u'_Y)^2 \dfrac{\partial u'_X}{\partial U_j}}{((1 + u'_X)^2 + (u'_Y)^2)^{\frac{3}{2}}} \tag{54}$$

$$\frac{\partial c_4}{\partial U_j} = \frac{(1 + u'_X)^2 \dfrac{\partial u'_Y}{\partial U_j} - (1 + u'_X) u'_Y \dfrac{\partial u'_X}{\partial U_j}}{((1 + u'_X)^2 + (u'_Y)^2)^{\frac{3}{2}}} \tag{55}$$

$$\begin{aligned}
\frac{\partial c_5}{\partial U_j} = \frac{1}{((1 + u'_X)^2 + (u'_Y)^2)^2} \Bigg\{ & (1 + u'_X)^3 \frac{\partial u''_Y}{\partial U_j} + (1 + u'_X)^2 \left(-u''_Y \frac{\partial u'_X}{\partial U_j} - u''_X \frac{\partial u'_Y}{\partial U_j} - u'_Y \frac{\partial u''_X}{\partial U_j} \right) \\
& + (1 + u'_X) \left(-2 u''_Y u'_Y \frac{\partial u'_Y}{\partial U_j} + 2 u'_Y u''_X \frac{\partial u'_X}{\partial U_j} + (u'_Y)^2 \frac{\partial u''_Y}{\partial U_j} \right) \\
& + (u'_Y)^2 \left(u''_X \frac{\partial u'_Y}{\partial U_j} + u''_Y \frac{\partial u'_X}{\partial U_j} \right) - (u'_Y)^3 \frac{\partial u''_X}{\partial U_j} \Bigg\}
\end{aligned} \tag{56}$$

5　変位の field-consistent な補間関数とその X に関する偏微分

5.1　変位 u_X、u_Y の補間関数

文献 [1] の field-consistent な補間関数を別の手順で導く。
梁要素の端点を 1、2 とし、中点を 3 としたとき、節点変位ベクトル $\boldsymbol{U}$ を次のように定義する。

$$\boldsymbol{U} = \begin{bmatrix} u_{X1} & u_{Y1} & \theta_{Z1} & u_{X2} & u_{Y2} & \theta_{Z2} & u_{X3} \end{bmatrix}^T \tag{57}$$

この節点値を用いて、変位 $\boldsymbol{u}$ を次のように補間する。

$$\boldsymbol{u} = \begin{bmatrix} u_X & u_Y \end{bmatrix}^T \tag{58}$$

$$u_X = N_1 u_{X1} + N_4 u_{X2} + N_7 u_{X3} \tag{59}$$

$$u_Y = N_2 u_{Y1} + N_3 u'_Y(0) + N_5 u_{Y2} + N_6 u'_Y(L) \tag{60}$$

$$
\begin{aligned}
\xi &= \frac{2X}{L} - 1 &(61)\\
L_1 &= \frac{1}{2}(1-\xi) &(62)\\
L_2 &= \frac{1}{2}(1+\xi) &(63)\\
N_1 &= L_1(2L_1 - 1)\\
&= \frac{1}{2}(1-\xi)(-\xi)\\
&= 1 - \frac{3}{L}X + \frac{2}{L^2}X^2 &(64)\\
N_4 &= L_2(2L_2 - 1)\\
&= \frac{1}{2}(1+\xi)\xi\\
&= -\frac{1}{L}X + \frac{2}{L^2}X^2 &(65)\\
N_7 &= 4L_1L_2\\
&= (1-\xi)(1+\xi)\\
&= \frac{4}{L}X - \frac{4}{L^2}X^2 &(66)\\
N_2 &= \frac{1}{4}(1-\xi)^2(2+\xi)\\
&= 1 - \frac{3}{L^2}X^2 + \frac{2}{L^3}X^3 &(67)\\
N_3 &= \frac{1}{8}L(1-\xi^2)(1-\xi)\\
&= X - \frac{2}{L}X^2 + \frac{1}{L^2}X^3 &(68)\\
N_5 &= \frac{1}{4}(1+\xi)^2(2-\xi)\\
&= \frac{3}{L^2}X^2 - \frac{2}{L^3}X^3 &(69)\\
N_6 &= \frac{1}{8}L(1+\xi)^2(\xi-1)\\
&= -\frac{X^2}{L} + \frac{X^3}{L^2} &(70)
\end{aligned}
$$

(5) 式の

$$
\tan\theta = \frac{u'_Y}{1+u'_X}
$$

より、

$$
\begin{aligned}
u'_Y(0) &= (1+u'_X(0))\tan\theta_{Z1} &(71)\\
u'_Y(L) &= (1+u'_X(L))\tan\theta_{Z2} &(72)
\end{aligned}
$$

$$u'_X = N'_1 u_{X1} + N'_4 u_{X2} + N'_7 u_{X3} \tag{73}$$
$$N'_1 = -\frac{3}{L} + \frac{4}{L^2}X \tag{74}$$
$$N'_4 = -\frac{1}{L} + \frac{4}{L^2}X \tag{75}$$
$$N'_7 = \frac{4}{L} - \frac{8}{L^2}X \tag{76}$$
$$N'_1(0) = -\frac{3}{L},\ N'_4(0) = -\frac{1}{L},\ N'_7(0) = \frac{4}{L} \tag{77}$$
$$N'_1(L) = \frac{1}{L},\ N'_4(L) = \frac{3}{L},\ N'_7(L) = -\frac{4}{L} \tag{78}$$
$$u'_X(0) = -\frac{3}{L}u_{X1} - \frac{1}{L}u_{X2} + \frac{4}{L}u_{X3} \tag{79}$$
$$u'_X(L) = \frac{1}{L}u_{X1} + \frac{3}{L}u_{X2} - \frac{4}{L}u_{X3} \tag{80}$$
$$u'_Y(0) = \left(1 - \frac{3}{L}u_{X1} - \frac{1}{L}u_{X2} + \frac{4}{L}u_{X3}\right)\tan\theta_{Z1} \tag{81}$$
$$u'_Y(L) = \left(1 + \frac{1}{L}u_{X1} + \frac{3}{L}u_{X2} - \frac{4}{L}u_{X3}\right)\tan\theta_{Z2} \tag{82}$$

以上から、u_X、u_Y を X の関数で表すと、

$$\begin{aligned}
u_X &= \left(1 - \frac{3}{L}X + \frac{2}{L^2}X^2\right)u_{X1} + \left(-\frac{1}{L}X + \frac{2}{L^2}X^2\right)u_{X2} + \left(\frac{4}{L}X - \frac{4}{L^2}X^2\right)u_{X3} \\
&= u_{X1} - \frac{X}{L}(3u_{X1} + u_{X2} - 4u_{X3}) + \frac{X^2}{L^2}2(u_{X1} + u_{X2} - 2u_{X3})
\end{aligned} \tag{83}$$

$$\begin{aligned}
u_Y &= \left(1 - \frac{3}{L^2}X^2 + \frac{2}{L^3}X^3\right)u_{Y1} \\
&\quad + \left(X - \frac{2}{L}X^2 + \frac{1}{L^2}X^3\right)\left(1 - \frac{3}{L}u_{X1} - \frac{1}{L}u_{X2} + \frac{4}{L}u_{X3}\right)\tan\theta_{Z1} \\
&\quad + \left(\frac{3X^2}{L^2} - \frac{2}{L^3}X^3\right)u_{Y2} \\
&\quad + \left(-\frac{X^2}{L} + \frac{X^3}{L^2}\right)\left(1 + \frac{1}{L}u_{X1} + \frac{3}{L}u_{X2} - \frac{4}{L}u_{X3}\right)\tan\theta_{Z2} \\
&= u_{Y1} + \frac{X}{L}[(L - 3u_{X1} - u_{X2} + 4u_{X3})\tan\theta_{Z1}] \\
&\qquad - \frac{X^2}{L^2}[3(u_{Y1} - u_{Y2}) \\
&\qquad\qquad + 2(L - 3u_{X1} - u_{X2} + 4u_{X3})\tan\theta_{Z1} \\
&\qquad\qquad + (L + u_{X1} + 3u_{X2} - 4u_{X3})\tan\theta_{Z2}] \\
&\qquad + \frac{X^3}{L^3}[2(u_{Y1} - u_{Y2}) \\
&\qquad\qquad + (L - 3u_{X1} - u_{X2} + 4u_{X3})\tan\theta_{Z1} \\
&\qquad\qquad + (L + u_{X1} + 3u_{X2} - 4u_{X3})\tan\theta_{Z2}]
\end{aligned} \tag{84}$$

(83) 式、(84) 式の u_X、u_Y が field-consistent な補間関数で文献 [1] と一致する。補間関数は節点値に関して線形にならないことが分かる。

6 運動エネルギーと慣性力

ここからは動的解析の定式化を行う。運動エネルギー K

$$K = \frac{1}{2}\rho\left\{\int_L A(\dot{u}_X^2 + \dot{u}_Y^2)dX + \int_L I_Z\dot{\theta}^2 dX\right\} \tag{85}$$

を用いると、Lagrange の運動方程式より慣性力 $\mathbf{\Phi}_K$ は次式で求まる。

$$\mathbf{\Phi}_K = \frac{d}{dt}\left[\frac{\partial K}{\partial \dot{\boldsymbol{U}}}\right] - \left[\frac{\partial K}{\partial \boldsymbol{U}}\right] \tag{86}$$

$$\dot{u}_X = \frac{\partial u_X}{\partial t} = \frac{\partial u_X}{\partial U_l}\frac{\partial U_l}{\partial t} = \boldsymbol{N}_X{}^T\dot{\boldsymbol{U}} \tag{87}$$

$$\dot{u}_Y = \boldsymbol{N}_Y^T\dot{\boldsymbol{U}} \tag{88}$$

$$\dot{\theta} = -\frac{u'_Y}{(1+u'_X)^2 + (u'_Y)^2}\frac{\partial u'_X}{\partial t} + \frac{1+u'_X}{(1+u'_X)^2 + (u'_Y)^2}\frac{\partial u'_Y}{\partial t} \tag{89}$$

((35) 式より)

$$= \left(-\frac{u'_Y}{(1+u'_X)^2 + (u'_Y)^2}\boldsymbol{N}'_X{}^T + \frac{1+u'_X}{(1+u'_X)^2 + (u'_Y)^2}\boldsymbol{N}'_Y{}^T\right)\frac{\partial \boldsymbol{U}}{\partial t}$$

$$= \boldsymbol{N}_\theta{}^T\dot{\boldsymbol{U}} \tag{90}$$

ここに、

$$\dot{\boldsymbol{U}} = \begin{bmatrix}\dot{u}_{X1} & \dot{u}_{Y1} & \dot{\theta}_{Z1} & \dot{u}_{X2} & \dot{u}_{Y2} & \dot{\theta}_{Z2} & \dot{u}_{X3}\end{bmatrix} \tag{91}$$

$$\boldsymbol{N}_X = \begin{bmatrix}\frac{\partial u_X}{\partial u_{X1}} & \frac{\partial u_X}{\partial u_{Y1}} & \frac{\partial u_X}{\partial u_{\theta 1}} & \frac{\partial u_X}{\partial u_{X2}} & \frac{\partial u_X}{\partial u_{Y2}} & \frac{\partial u_X}{\partial u_{\theta 2}} & \frac{\partial u_X}{\partial u_{X3}}\end{bmatrix} \tag{92}$$

$$\boldsymbol{N}_Y = \begin{bmatrix}\frac{\partial u_Y}{\partial u_{X1}} & \frac{\partial u_Y}{\partial u_{Y1}} & \frac{\partial u_Y}{\partial u_{\theta 1}} & \frac{\partial u_Y}{\partial u_{X2}} & \frac{\partial u_Y}{\partial u_{Y2}} & \frac{\partial u_Y}{\partial u_{\theta 2}} & \frac{\partial u_Y}{\partial u_{X3}}\end{bmatrix} \tag{93}$$

$$\boldsymbol{N}_\theta = -\frac{u'_Y}{(1+u'_X)^2 + (u'_Y)^2}\boldsymbol{N}'_X{}^T + \frac{1+u'_X}{(1+u'_X)^2 + (u'_Y)^2}\boldsymbol{N}'_Y{}^T \tag{94}$$

$$\frac{1}{2}\rho\int_L A\dot{u}_X^2 dX = \frac{1}{2}\dot{\boldsymbol{U}}^T\boldsymbol{M}_X\dot{\boldsymbol{U}} \tag{95}$$

$$\frac{1}{2}\rho\int_L A\dot{u}_Y^2 dX = \frac{1}{2}\dot{\boldsymbol{U}}^T\boldsymbol{M}_Y\dot{\boldsymbol{U}} \tag{96}$$

$$\frac{1}{2}\rho\int_L I_Z\dot{\theta}^2 dX = \frac{1}{2}\dot{\boldsymbol{U}}^T\boldsymbol{M}_\theta\dot{\boldsymbol{U}} \tag{97}$$

$$\boldsymbol{M}_X = \rho A\int_L \boldsymbol{N}_X\boldsymbol{N}_X^T dX \tag{98}$$

$$\boldsymbol{M}_Y = \rho A\int_L \boldsymbol{N}_X\boldsymbol{N}_Y^T dX \tag{99}$$

$$\boldsymbol{M}_\theta = \rho I_Z\int_L \boldsymbol{N}_\theta\boldsymbol{N}_\theta^T dX \tag{100}$$

$$(\boldsymbol{M}_X)ij = \rho A\int_L N_{Xi}N_{Xj} dX \tag{101}$$

$$(\boldsymbol{M}_Y)_{ij} = \rho A\int_L N_{Yi}N_{Yj} dX \tag{102}$$

$$(\boldsymbol{M}_\theta)_{ij} = \rho I_Z\int_L N_{\theta i}N_{\theta j} dX \tag{103}$$

したがって、

$$K = \frac{1}{2}\dot{\boldsymbol{U}}^T\boldsymbol{M}\dot{\boldsymbol{U}} \tag{104}$$

$$\boldsymbol{M} = \boldsymbol{M}_X + \boldsymbol{M}_Y + \boldsymbol{M}_\theta \tag{105}$$

(86) 式

$$\boldsymbol{\Phi}_K = \frac{d}{dt}\left[\frac{\partial K}{\partial\dot{\boldsymbol{U}}}\right] - \left[\frac{\partial K}{\partial\boldsymbol{U}}\right] \tag{106}$$

を計算する。

(104) 式より、

$$\frac{\partial K}{\partial\dot{\boldsymbol{U}}} = \boldsymbol{M} \tag{107}$$

$$\frac{d}{dt}\left[\frac{\partial K}{\partial\dot{\boldsymbol{U}}}\right] = \boldsymbol{M}\ddot{\boldsymbol{U}} + \dot{\boldsymbol{M}}\dot{\boldsymbol{U}} \tag{108}$$

$\dot{\boldsymbol{M}}$ を求める。

$$\dot{\boldsymbol{M}} = \frac{\partial\boldsymbol{M}}{\partial t} = \frac{\partial\boldsymbol{M}}{\partial U_l}\frac{\partial U_l}{\partial t} = \frac{\partial\boldsymbol{M}}{\partial U_l}\dot{U}_l$$

$$(\dot{\boldsymbol{M}})_{ij} = \frac{\partial M_{ij}}{\partial U_l}\dot{U}_l \tag{109}$$

また、

$$\frac{\partial K}{\partial\boldsymbol{U}} = \frac{1}{2}\dot{\boldsymbol{U}}^T\frac{\partial\boldsymbol{M}}{\partial U_i}\dot{\boldsymbol{U}}$$

$$\left(\frac{\partial K}{\partial\boldsymbol{U}}\right)_i = \frac{1}{2}\dot{U}_l\frac{\partial M_{lm}}{\partial U_i}\dot{U}_m \tag{110}$$

したがって、

$$
\begin{aligned}
(\boldsymbol{\Phi}_K)_i &= M_{im}\ddot{U}_m + \dot{M}_{im}\dot{U}_m - \frac{1}{2}\dot{U}_l\frac{\partial M_{lm}}{\partial U_i}\dot{U}_m \\
&= M_{im}\ddot{U}_m + \left(\frac{\partial M_{im}}{\partial U_l}\dot{U}_l\right)\dot{U}_m - \frac{1}{2}\dot{U}_l\frac{\partial M_{lm}}{\partial U_i}\dot{U}_m
\end{aligned} \tag{111}
$$

$\dfrac{\partial \boldsymbol{M}}{\partial \boldsymbol{U}}$ を求める。

$$
\frac{\partial M_{Xij}}{\partial U_l} = \rho A\int_L \frac{\partial N_{Xi}}{\partial U_l}N_{Xj} + N_{Xi}\frac{\partial N_{Xj}}{\partial U_l}dX \tag{112}
$$

$$
\frac{\partial M_{Yij}}{\partial U_l} = \rho A\int_L \frac{\partial N_{Yi}}{\partial U_l}N_{Yj} + N_{Yi}\frac{\partial N_{Yj}}{\partial U_l}dX \tag{113}
$$

$$
\frac{\partial M_{\theta ij}}{\partial U_l} = \rho I_Z\int_L \frac{\partial N_{\theta i}}{\partial U_l}N_{\theta j} + N_{\theta i}\frac{\partial N_{\theta j}}{\partial U_l}dX \tag{114}
$$

$$
\frac{\partial M_{ij}}{\partial U_l} = \frac{\partial M_{Xij}}{\partial U_l} + \frac{\partial M_{Yij}}{\partial U_l} + \frac{\partial M_{\theta ij}}{\partial U_l} \tag{115}
$$

7　非線形運動方程式

外力 $\boldsymbol{f}$ を省略すると、非線形運動方程式は、

$$
\boldsymbol{\Phi}_K + \boldsymbol{\Phi} = 0 \tag{116}
$$

ここに、Φ_K：慣性力、Φ：弾性力である。

$$
\boldsymbol{K} = \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{U}} \tag{117}
$$

$$
\boldsymbol{M} = \frac{\partial \boldsymbol{\Phi}_K}{\partial \ddot{\boldsymbol{U}}} \tag{118}
$$

$$
\boldsymbol{C}_K = \frac{\partial \boldsymbol{\Phi}_K}{\partial \dot{\boldsymbol{U}}} \tag{119}
$$

$$
\boldsymbol{K}_K = \frac{\partial \boldsymbol{\Phi}_K}{\partial \boldsymbol{U}} \tag{120}
$$

$\boldsymbol{K}$、$\boldsymbol{M}$ はすでに求めた。

$\boldsymbol{C}_K$ を求める。

$$
\begin{aligned}
\boldsymbol{C}_K &= \frac{\partial \boldsymbol{\Phi}_K}{\partial \dot{\boldsymbol{U}}} \\
&= \dot{\boldsymbol{M}} + \boldsymbol{C}_1 - \boldsymbol{C}_1^T
\end{aligned} \tag{121}
$$

$$
(\boldsymbol{C}_1)_{ij} = \frac{\partial M_{im}}{\partial U_j}\dot{U}_m \tag{122}
$$

$\boldsymbol{K}_K$ を求める。

$$
\begin{aligned}
\boldsymbol{K}_K &= \frac{\partial \boldsymbol{\Phi}_K}{\partial \boldsymbol{U}} \\
&= \boldsymbol{K}_1 + \boldsymbol{K}_2 - \boldsymbol{K}_3 && (123)\\
(\boldsymbol{K}_1)_{ij} &= \frac{\partial M_{im}}{\partial U_j}\ddot{U}_m && (124)\\
(\boldsymbol{K}_2)_{ij} &= \left(\frac{\partial^2 M_{im}}{\partial U_j \partial U_l}\dot{U}_l\right)\dot{U}_m && (125)\\
(\boldsymbol{K}_3)_{ij} &= \frac{1}{2}\dot{U}_l \frac{\partial^2 M_{lm}}{\partial U_i}\partial U_j \dot{U}_m && (126)
\end{aligned}
$$

非線形運動方程式に Newton-Raphson 法を適用して線形化すると、

$$
\begin{aligned}
\boldsymbol{K}\Delta\boldsymbol{U} + \boldsymbol{M}\Delta\ddot{\boldsymbol{U}} + \boldsymbol{C}_K\Delta\dot{\boldsymbol{U}} + \boldsymbol{K}_K\Delta\boldsymbol{U} &= -\boldsymbol{\Phi}_K - \boldsymbol{\Phi} && (127)\\
\Delta\boldsymbol{U} &= \boldsymbol{U} - \boldsymbol{U}^{n-1} && (128)\\
\Delta\dot{\boldsymbol{U}} &= \dot{\boldsymbol{U}} - \dot{\boldsymbol{U}}^{n-1} && (129)\\
\Delta\ddot{\boldsymbol{U}} &= \ddot{\boldsymbol{U}} - \ddot{\boldsymbol{U}}^{n-1} && (130)
\end{aligned}
$$

Newmark β 法

$$
\begin{aligned}
\dot{\boldsymbol{U}} &= \frac{1}{\beta\Delta t^2}(\boldsymbol{U} - \boldsymbol{U}^{t-1}) - \frac{1}{\beta\Delta t}\dot{\boldsymbol{U}}^{t-1} - \left(\frac{1}{2\beta} - 1\right)\ddot{\boldsymbol{U}}^{t-1}\\
\ddot{\boldsymbol{U}} &= \frac{\gamma}{\beta\Delta t}(\boldsymbol{U} - \boldsymbol{U}^{t-1}) + \left(1 - \frac{\gamma}{\beta}\right)\dot{\boldsymbol{U}}^{t-1} + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{\boldsymbol{U}}^{t-1}
\end{aligned}
$$

を適用すると、

$$
\begin{aligned}
&\boldsymbol{K}\boldsymbol{U} + \frac{1}{\beta\Delta t^2}\boldsymbol{M}\boldsymbol{U} + \frac{\gamma}{\beta\Delta t}\boldsymbol{C}_K\boldsymbol{U} + \boldsymbol{K}_K\boldsymbol{U}\\
&\quad = \boldsymbol{K}\boldsymbol{U}^{n-1} + \boldsymbol{M}\ddot{\boldsymbol{U}}^{n-1} + \boldsymbol{C}_K\dot{\boldsymbol{U}}^{n-1} + \boldsymbol{K}_K\boldsymbol{U}^{n-1}\\
&\quad + \boldsymbol{M}\left\{\frac{1}{\beta\Delta t^2}\boldsymbol{U}^{t-1} + \frac{1}{\beta\Delta t}\dot{\boldsymbol{U}}^{t-1} + \left(\frac{1}{2\beta} - 1\right)\ddot{\boldsymbol{U}}^{t-1}\right\}\\
&\quad + \boldsymbol{C}_K\left\{\frac{\gamma}{\beta\Delta t}\boldsymbol{U}^{t-1} + \left(\frac{\gamma}{\beta} - 1\right)\dot{\boldsymbol{U}}^{t-1} + \left(\frac{\gamma}{2\beta} - 1\right)\Delta t\ddot{\boldsymbol{U}}^{t-1}\right\} && (131)
\end{aligned}
$$

8 グローバル座標系への変換

グローバル座標系では、参照面 C_0 は X 軸に沿わず、C_0 のローカル座標系の $\overline{X}$ は、グローバル座標系の X に対して角度 β_0 をなす。

$$
\begin{aligned}
\cos\beta_0 &= \frac{X_{21}}{L_0} \\
\sin\beta_0 &= \frac{Y_{21}}{L_0} \qquad (132) \\
X_{21} &= X_2 - X_1 \\
Y_{21} &= Y_2 - Y_1 \\
L_0 &= \sqrt{X_{21}^2 + Y_{21}^2} \qquad (133)
\end{aligned}
$$

$\overline{\boldsymbol{U}}$ をローカルな節点値ベクトル、$\boldsymbol{U}$ をグローバルな節点値ベクトルとすると、

$$
\begin{aligned}
\overline{\boldsymbol{U}} &= \boldsymbol{T}\boldsymbol{U} \\
\boldsymbol{U} &= \boldsymbol{T}^T\overline{\boldsymbol{U}} \qquad (134)
\end{aligned}
$$

$\boldsymbol{T}$ は直交行列でなければならない。

$\overline{u}_{X3}$ は独立した節点値なのでローカルの値のままとすると、

$$
\overline{\boldsymbol{U}} = \begin{bmatrix} \overline{u_{X1}} & \overline{u_{Y1}} & \overline{\theta_{Z1}} & \overline{u_{X2}} & \overline{u_{Y2}} & \overline{\theta_{Z2}} & \overline{u_{X3}} \end{bmatrix}^T \qquad (135)
$$

$$
\overline{\boldsymbol{U}} = \begin{bmatrix} u_{X1} & u_{Y1} & \theta_{Z1} & u_{X2} & u_{Y2} & \theta_{Z2} & u_{X3} \end{bmatrix}^T \qquad (136)
$$

$$
\boldsymbol{T} = \begin{bmatrix}
\cos\beta_0 & \sin\beta_0 & 0 & 0 & 0 & 0 & 0 \\
-\sin\beta_0 & \cos\beta_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos\beta_0 & \sin\beta_0 & 0 & 0 \\
0 & 0 & 0 & -\sin\beta_0 & \cos\beta_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \qquad (137)
$$

グローバル座標系の剛性方程式への変換

$$
\begin{aligned}
& \overline{\boldsymbol{K}}\Delta\overline{\boldsymbol{U}} + \overline{\boldsymbol{M}}\Delta\ddot{\overline{\boldsymbol{U}}} + \overline{\boldsymbol{C}}_K\Delta\dot{\overline{\boldsymbol{U}}} + \overline{\boldsymbol{K}}_K\Delta\overline{\boldsymbol{U}} = \overline{\boldsymbol{f}} - \overline{\boldsymbol{\Phi}} \qquad (138) \\
\rightarrow \quad & \boldsymbol{T}^T\overline{\boldsymbol{K}}\boldsymbol{T}\Delta\boldsymbol{U} + \boldsymbol{T}^T\overline{\boldsymbol{M}}\boldsymbol{T}\Delta\ddot{\boldsymbol{U}} + \boldsymbol{T}^T\overline{\boldsymbol{C}}_K\boldsymbol{T}\Delta\dot{\boldsymbol{U}} + \boldsymbol{T}^T\overline{\boldsymbol{K}}_K\boldsymbol{T}\Delta\boldsymbol{U} = \boldsymbol{T}^T\overline{\boldsymbol{f}} - \boldsymbol{T}^T\overline{\boldsymbol{\Phi}} \\
\rightarrow \quad & \boldsymbol{K}\Delta\boldsymbol{U} + \boldsymbol{M}\Delta\ddot{\boldsymbol{U}} + \boldsymbol{C}_K\Delta\dot{\boldsymbol{U}} + \boldsymbol{K}_K\Delta\boldsymbol{U} = \boldsymbol{f} - \boldsymbol{\Phi} \qquad (139)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{K} &= \boldsymbol{T}^T\overline{\boldsymbol{K}}\boldsymbol{T} \qquad (140) \\
\boldsymbol{M} &= \boldsymbol{T}^T\overline{\boldsymbol{M}}\boldsymbol{T} \qquad (141) \\
\boldsymbol{C}_K &= \boldsymbol{T}^T\overline{\boldsymbol{C}}_K\boldsymbol{T} \qquad (142) \\
\boldsymbol{K}_K &= \boldsymbol{T}^T\overline{\boldsymbol{K}}_K\boldsymbol{T} \qquad (143) \\
\boldsymbol{f} &= \boldsymbol{T}^T\overline{\boldsymbol{f}} \qquad (144) \\
\boldsymbol{\Phi} &= \boldsymbol{T}^T\overline{\boldsymbol{\Phi}} \qquad (145)
\end{aligned}
$$

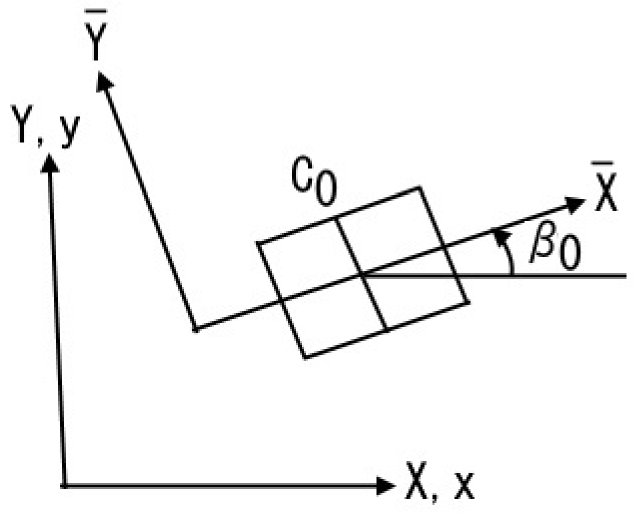

図 2　Global 座標系

9　変位 u_X、u_Y とその微分

9.1　u_X、u_Y

すでに求めた ((83) 式、(84) 式)。

9.2　u'_X、u'_Y

$$u'_X = -\frac{1}{L}(3u_{X1} + u_{X2} - 4u_{X3}) + \frac{X}{L^3}4(u_{X1} + u_{X2} - 2u_{X3}) \tag{146}$$

$$\begin{aligned}
u'_Y = &\frac{1}{L}[(L - 3u_{X1} - u_{X2} + 4u_{X3})\tan\theta_{Z1}] \\
&-\frac{X}{L^2}[6(u_{Y1} - u_{Y2}) \\
&\qquad +4(L - 3u_{X1} - u_{X2} + 4u_{X3})\tan\theta_{Z1} \\
&\qquad +2(L + u_{X1} + 3u_{X2} - 4u_{X3})\tan\theta_{Z2}] \\
&+\frac{X^2}{L^3}[6(u_{Y1} - u_{Y2}) \\
&\qquad +3(L - 3u_{X1} - u_{X2} + 4u_{X3})\tan\theta_{Z1} \\
&\qquad +3(L + u_{X1} + 3u_{X2} - 4u_{X3})\tan\theta_{Z2}] \qquad (147)
\end{aligned}$$

(148)

9.3　u''_X、u''_Y

$$u''_X = \frac{1}{L^2}4(u_{X1} + u_{X2} - 2u_{X3}) \tag{149}$$

$$\begin{aligned}
u''_Y = &-\frac{1}{L^2}[6(u_{Y1} - u_{Y2}) \\
&\qquad +4(L - 3u_{X1} - u_{X2} + 4u_{X3})\tan\theta_{Z1} \\
&\qquad +2(L + u_{X1} + 3u_{X2} - 4u_{X3})\tan\theta_{Z2}] \\
&+\frac{X}{L^3}[12(u_{Y1} - u_{Y2}) \\
&\qquad +6(L - 3u_{X1} - u_{X2} + 4u_{X3})\tan\theta_{Z1} \\
&\qquad +6(L + u_{X1} + 3u_{X2} - 4u_{X3})\tan\theta_{Z2}] \qquad (150)
\end{aligned}$$

9.4 $\dfrac{\partial u'_X}{\partial U_j}$、$\dfrac{\partial u'_Y}{\partial U_j}$

$\dfrac{\partial u'_X}{\partial U_j}$ を求める。

$$\frac{\partial u'_X}{\partial u_{X1}} = -\frac{1}{L}3 + \frac{X}{L^2}4 \tag{151}$$

$$\frac{\partial u'_X}{\partial u_{Y1}} = 0 \tag{152}$$

$$\frac{\partial u'_X}{\partial \theta_{Z1}} = 0 \tag{153}$$

$$\frac{\partial u'_X}{\partial u_{X2}} = -\frac{1}{L} + \frac{X}{L^2}4 \tag{154}$$

$$\frac{\partial u'_X}{\partial u_{Y2}} = 0 \tag{155}$$

$$\frac{\partial u'_X}{\partial \theta_{Z2}} = 0 \tag{156}$$

$$\frac{\partial u'_X}{\partial u_{X3}} = \frac{1}{L}4 - \frac{X}{L^2}8 \tag{157}$$

$\dfrac{\partial u'_Y}{\partial U_j}$ を求める。

$$\frac{\partial u'_Y}{\partial u_{X1}} = -\frac{1}{L}3\tan\theta_{Z1} + \frac{X}{L^2}(12\tan\theta_{Z1} - 2\tan\theta_{Z2}) - \frac{X^2}{L^3}(9\tan\theta_{Z1} - 3\tan\theta_{Z2}) \tag{158}$$

$$\frac{\partial u'_Y}{\partial u_{Y1}} = -\frac{X}{L^2}6 + \frac{X^2}{L^3}6 \tag{159}$$

$$\frac{\partial u'_Y}{\partial \theta_{Z1}} = \frac{1}{L}\left[(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{1}{\cos^2\theta_{Z1}}\right] - \frac{X}{L^2}\left[4(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{1}{\cos^2\theta_{Z1}}\right] + \frac{X^2}{L^3}\left[3(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{1}{\cos^2\theta_{Z1}}\right] \tag{160}$$

$$\frac{\partial u'_Y}{\partial u_{X2}} = -\frac{1}{L}\tan\theta_{Z1} + \frac{X}{L^2}(4\tan\theta_{Z1} - 6\tan\theta_{Z2}) - \frac{X^2}{L^3}(3\tan\theta_{Z1} - 9\tan\theta_{Z2}) \tag{161}$$

$$\frac{\partial u'_Y}{\partial u_{Y2}} = \frac{X}{L^2}6 - \frac{X^2}{L^3}6 \tag{162}$$

$$\frac{\partial u'_Y}{\partial \theta_{Z2}} = -\frac{X}{L^2}\left[2(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{1}{\cos^2\theta_{Z2}}\right] + \frac{X^2}{L^3}\left[3(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{1}{\cos^2\theta_{Z2}}\right] \tag{163}$$

$$\frac{\partial u'_Y}{\partial u_{X3}} = \frac{1}{L}4\tan\theta_{Z1} - \frac{X}{L^2}(16\tan\theta_{Z1} - 8\tan\theta_{Z2}) + \frac{X^2}{L^3}(12\tan\theta_{Z1} - 12\tan\theta_{Z2}) \tag{164}$$

9.5 $\dfrac{\partial u''_X}{\partial U_j}$、$\dfrac{\partial u''_Y}{\partial U_j}$

$\dfrac{\partial u''_X}{\partial U_j}$ を求める。

$$\frac{\partial u''_X}{\partial u_{X1}} = \frac{1}{L^2}4 \tag{165}$$
$$\frac{\partial u''_X}{\partial u_{Y1}} = 0 \tag{166}$$
$$\frac{\partial u''_X}{\partial \theta_{Z1}} = 0 \tag{167}$$
$$\frac{\partial u''_X}{\partial u_{X2}} = \frac{1}{L^2}4 \tag{168}$$
$$\frac{\partial u''_X}{\partial u_{Y2}} = 0 \tag{169}$$
$$\frac{\partial u''_X}{\partial \theta_{Z2}} = 0 \tag{170}$$
$$\frac{\partial u''_X}{\partial u_{X3}} = -\frac{1}{L^2}8 \tag{171}$$

$\dfrac{\partial u''_Y}{\partial U_j}$ を求める。

$$\frac{\partial u''_Y}{\partial u_{X1}} = \frac{1}{L^2}(12\tan\theta_{Z1} - 2\tan\theta_{Z2}) - \frac{X}{L^3}(18\tan\theta_{Z1} - 6\tan\theta_{Z2}) \tag{172}$$
$$\frac{\partial u''_Y}{\partial u_{Y1}} = -\frac{1}{L^2}6 + \frac{X}{L^3}12 \tag{173}$$
$$\begin{aligned}\frac{\partial u''_Y}{\partial \theta_{Z1}} &= -\frac{1}{L^2}\left[4(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{1}{\cos^2\theta_{Z1}}\right] \\ &\quad + \frac{X}{L^3}\left[6(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{1}{\cos^2\theta_{Z1}}\right]\end{aligned} \tag{174}$$
$$\frac{\partial u''_Y}{\partial u_{X2}} = \frac{1}{L^2}(4\tan\theta_{Z1} - 6\tan\theta_{Z2}) - \frac{X}{L^3}(6\tan\theta_{Z1} - 18\tan\theta_{Z2}) \tag{175}$$
$$\frac{\partial u''_Y}{\partial u_{Y2}} = \frac{1}{L^2}6 - \frac{X}{L^3}12 \tag{176}$$
$$\begin{aligned}\frac{\partial u''_Y}{\partial \theta_{Z2}} &= -\frac{1}{L^2}\left[2(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{1}{\cos^2\theta_{Z2}}\right] \\ &\quad + \frac{X}{L^3}\left[6(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{1}{\cos^2\theta_{Z2}}\right]\end{aligned} \tag{177}$$
$$\frac{\partial u''_Y}{\partial u_{X3}} = -\frac{1}{L^2}(16\tan\theta_{Z1} - 8\tan\theta_{Z2}) + \frac{X}{L^3}(24\tan\theta_{Z1} - 24\tan\theta_{Z2}) \tag{178}$$

9.6 $\dfrac{\partial^2 u'_X}{\partial U_l \partial U_m}$、$\dfrac{\partial^2 u'_Y}{\partial U_l \partial U_m}$

$\dfrac{\partial^2 u'_X}{\partial U_l \partial U_m}$ を求める。

$$\frac{\partial^2 u'_X}{\partial U_l \partial U_m} = 0 \tag{179}$$

$\dfrac{\partial^2 u'_Y}{\partial U_l \partial U_m}$ を求める。非 0 項のみ示す。

$$\frac{\partial^2 u'_Y}{\partial u_{X1} \partial \theta_{Z1}} = -\frac{1}{L}\frac{3}{\cos^2\theta_{Z1}} + \frac{X}{L^2}\frac{12}{\cos^2\theta_{Z1}} - \frac{X^2}{L^3}\frac{9}{\cos^2\theta_{Z1}} \tag{180}$$

$$\frac{\partial^2 u'_Y}{\partial u_{X1} \partial \theta_{Z2}} = -\frac{X}{L^2}\frac{2}{\cos^2\theta_{Z2}} + \frac{X^2}{L^3}\frac{3}{\cos^2\theta_{Z2}} \tag{181}$$

$$\frac{\partial^2 u'_Y}{\partial \theta_{Z1} \partial u_{X1}} = -\frac{1}{L}\frac{3}{\cos^2\theta_{Z1}} + \frac{X}{L^2}\frac{12}{\cos^2\theta_{Z1}} - \frac{X^2}{L^3}\frac{9}{\cos^2\theta_{Z1}} \tag{182}$$

$$\begin{aligned}\frac{\partial^2 u'_Y}{\partial {\theta_{Z1}}^2} = & \frac{1}{L}\left[2(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}}\right] \\ & -\frac{X}{L^2}\left[8(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}}\right] \\ & +\frac{X^2}{L^3}\left[6(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}}\right]\end{aligned} \tag{183}$$

$$\frac{\partial^2 u'_Y}{\partial \theta_{Z1} \partial u_{X2}} = -\frac{1}{L}\frac{1}{\cos^2\theta_{Z1}} + \frac{X}{L^2}\frac{4}{\cos^2\theta_{Z1}} - \frac{X^2}{L^3}\frac{3}{\cos^2\theta_{Z1}} \tag{184}$$

$$\frac{\partial^2 u'_Y}{\partial \theta_{Z1} \partial u_{X3}} = \frac{1}{L}\frac{4}{\cos^2\theta_{Z1}} - \frac{X}{L^2}\frac{16}{\cos^2\theta_{Z1}} + \frac{X^2}{L^3}\frac{12}{\cos^2\theta_{Z1}} \tag{185}$$

$$\frac{\partial^2 u'_Y}{\partial u_{X2} \partial \theta_{Z1}} = -\frac{1}{L}\frac{1}{\cos^2\theta_{Z1}} + \frac{X}{L^2}\frac{4}{\cos^2\theta_{Z1}} - \frac{X^2}{L^3}\frac{3}{\cos^2\theta_{Z1}} \tag{186}$$

$$\frac{\partial^2 u'_Y}{\partial u_{X2} \partial \theta_{Z2}} = -\frac{X}{L^2}\frac{6}{\cos^2\theta_{Z2}} + \frac{X^2}{L^3}\frac{9}{\cos^2\theta_{Z2}} \tag{187}$$

$$\frac{\partial^2 u'_Y}{\partial \theta_{Z2} \partial u_{X1}} = -\frac{X}{L^2}\frac{2}{\cos^2\theta_{Z2}} + \frac{X^2}{L^3}\frac{3}{\cos^2\theta_{Z2}} \tag{188}$$

$$\frac{\partial^2 u'_Y}{\partial \theta_{Z2} \partial u_{X2}} = -\frac{X}{L^2}\frac{6}{\cos^2\theta_{Z2}} + \frac{X^2}{L^3}\frac{9}{\cos^2\theta_{Z2}} \tag{189}$$

$$\begin{aligned}\frac{\partial^2 u'_Y}{\partial {\theta_{Z2}}^2} &= -\frac{X}{L^2}\left[4(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}}\right] \\ &+\frac{X^2}{L^3}\left[6(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}}\right]\end{aligned} \tag{190}$$

$$\frac{\partial^2 u'_Y}{\partial \theta_{Z2} \partial u_{X3}} = \frac{X}{L^2}\frac{8}{\cos^2\theta_{Z2}} - \frac{X^2}{L^3}\frac{12}{\cos^2\theta_{Z2}} \tag{191}$$

$$\frac{\partial^2 u'_Y}{\partial u_{X3} \partial \theta_{Z1}} = \frac{1}{L}\frac{4}{\cos^2\theta_{Z1}} - \frac{X}{L^2}\frac{16}{\cos^2\theta_{Z1}} + \frac{X^2}{L^3}\frac{12}{\cos^2\theta_{Z1}} \tag{192}$$

$$\frac{\partial^2 u'_Y}{\partial u_{X3} \partial \theta_{Z2}} = \frac{X}{L^2}\frac{8}{\cos^2\theta_{Z2}} - \frac{X^2}{L^3}\frac{12}{\cos^2\theta_{Z2}} \tag{193}$$

10　形状関数に対応する $\boldsymbol{N}_X$、$\boldsymbol{N}_Y$ とその微分

10.1　$\boldsymbol{N}_X$、$\boldsymbol{N}_Y$

(31) 式、(32) 式

$$\boldsymbol{N}_X = \begin{bmatrix} \frac{\partial u_X}{\partial u_{X1}} & \frac{\partial u_X}{\partial u_{Y1}} & \frac{\partial u_X}{\partial \theta_{Z1}} & \frac{\partial u_X}{\partial u_{X2}} & \frac{\partial u_X}{\partial u_{Y2}} & \frac{\partial u_X}{\partial \theta_{Z2}} & \frac{\partial u_X}{\partial u_{X3}} \end{bmatrix}^T \tag{194}$$

$$\boldsymbol{N}_Y = \begin{bmatrix} \frac{\partial u_Y}{\partial u_{X1}} & \frac{\partial u_Y}{\partial u_{Y1}} & \frac{\partial u_Y}{\partial \theta_{Z1}} & \frac{\partial u_Y}{\partial u_{X2}} & \frac{\partial u_Y}{\partial u_{Y2}} & \frac{\partial u_Y}{\partial \theta_{Z2}} & \frac{\partial u_Y}{\partial u_{X3}} \end{bmatrix}^T \tag{195}$$

$$N_{X1} = \frac{\partial u_X}{\partial u_{X1}} = 1 - 3\frac{X}{L} + 2\frac{X^2}{L^2} \tag{196}$$

$$N_{X2} = \frac{\partial u_X}{\partial u_{Y1}} = 0 \tag{197}$$

$$N_{X3} = \frac{\partial u_X}{\partial \theta_{Z1}} = 0 \tag{198}$$

$$N_{X4} = \frac{\partial u_X}{\partial u_{X2}} = -\frac{X}{L} + 2\frac{X^2}{L^2} \tag{199}$$

$$N_{X5} = \frac{\partial u_X}{\partial u_{Y2}} = 0 \tag{200}$$

$$N_{X6} = \frac{\partial u_X}{\partial \theta_{Z2}} = 0 \tag{201}$$

$$N_{X7} = \frac{\partial u_X}{\partial u_{X3}} = 4\frac{X}{L} - 4\frac{X^2}{L^2} \tag{202}$$

$$\begin{aligned} N_{Y1} &= \frac{\partial u_Y}{\partial u_{X1}} \\ &= -\frac{X}{L}(3\tan\theta_{Z1}) + \frac{X^2}{L^2}(6\tan\theta_{Z1} - \tan\theta_{Z2}) - \frac{X^3}{L^3}(3\tan\theta_{Z1} - \tan\theta_{Z2}) \end{aligned} \tag{203}$$

$$N_{Y2} = \frac{\partial u_Y}{\partial u_{Y1}} = 1 - \frac{X^2}{L^2}3 + \frac{X^3}{L^3}2 \tag{204}$$

$$\begin{aligned} N_{Y3} &= \frac{\partial u_Y}{\partial \theta_{Z1}} \\ &= \frac{X}{L}\left[(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{1}{\cos^2\theta_{Z1}}\right] \\ &\quad -\frac{X^2}{L^2}\left[2(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{1}{\cos^2\theta_{Z1}}\right] \\ &\quad +\frac{X^3}{L^3}\left[(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{1}{\cos^2\theta_{Z1}}\right] \end{aligned} \tag{205}$$

$$\begin{aligned} N_{Y4} &= \frac{\partial u_Y}{\partial u_{X2}} \\ &= -\frac{X}{L}\tan\theta_{Z1} + \frac{X^2}{L^2}(2\tan\theta_{Z1} - 3\tan\theta_{Z2}) - \frac{X^3}{L^3}(\tan\theta_{Z1} - 3\tan\theta_{Z2}) \end{aligned} \tag{206}$$

$$N_{Y5} = \frac{\partial u_Y}{\partial u_{Y2}} = \frac{X^2}{L^2}3 - \frac{X^3}{L^3}2 \tag{207}$$

$$\begin{aligned} N_{Y6} &= \frac{\partial u_Y}{\partial \theta_{Z2}} \\ &= -\frac{X^2}{L^2}\left[(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{1}{\cos^2\theta_{Z2}}\right] \\ &\quad +\frac{X^3}{L^3}\left[(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{1}{\cos^2\theta_{Z2}}\right] \end{aligned} \tag{208}$$

$$\begin{aligned} N_{Y7} &= \frac{\partial u_Y}{\partial u_{X3}} \\ &= \frac{X}{L}(4\tan\theta_{Z1}) - \frac{X^2}{L^2}(8\tan\theta_{Z1} - 4\tan\theta_{Z2}) + \frac{X^3}{L^3}(4\tan\theta_{Z1} - 4\tan\theta_{Z2}) \end{aligned} \tag{209}$$

10.2 $\boldsymbol{N}'_X$、$\boldsymbol{N}'_Y$

$\boldsymbol{N}'_X$、$\boldsymbol{N}'_Y$ を求める。

$$N'_{X1} = -3\frac{1}{L} + 4\frac{X}{L^2} \tag{210}$$
$$N'_{X2} = 0 \tag{211}$$
$$N'_{X3} = 0 \tag{212}$$
$$N'_{X4} = -\frac{1}{L} + 4\frac{X}{L^2} \tag{213}$$
$$N'_{X5} = 0 \tag{214}$$
$$N'_{X6} = 0 \tag{215}$$
$$N'_{X7} = 4\frac{1}{L} - 8\frac{X}{L^2} \tag{216}$$

$$N'_{Y1} = -\frac{1}{L}(3\tan\theta_{Z1}) + \frac{X}{L^2}(12\tan\theta_{Z1} - 2\tan\theta_{Z2}) - \frac{X^2}{L^3}(9\tan\theta_{Z1} - 3\tan\theta_{Z2}) \tag{217}$$
$$N'_{Y2} = -\frac{X}{L^2}6 + \frac{X^2}{L^3}6 \tag{218}$$
$$\begin{aligned} N'_{Y3} = &\frac{1}{L}\left[(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{1}{\cos^2\theta_{Z1}}\right] \\ &-\frac{X}{L^2}\left[4(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{1}{\cos^2\theta_{Z1}}\right] \\ &+\frac{X^2}{L^3}\left[3(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{1}{\cos^2\theta_{Z1}}\right] \end{aligned} \tag{219}$$
$$N'_{Y4} = -\frac{1}{L}\tan\theta_{Z1} + \frac{X}{L^2}(4\tan\theta_{Z1} - 6\tan\theta_{Z2}) - \frac{X^2}{L^3}(3\tan\theta_{Z1} - 9\tan\theta_{Z2}) \tag{220}$$
$$N'_{Y5} = \frac{X}{L^2}6 - \frac{X^2}{L^3}6 \tag{221}$$
$$\begin{aligned} N'_{Y6} = &-\frac{X}{L^2}\left[2(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{1}{\cos^2\theta_{Z2}}\right] \\ &+\frac{X^2}{L^3}\left[3(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{1}{\cos^2\theta_{Z2}}\right] \end{aligned} \tag{222}$$
$$N'_{Y7} = \frac{1}{L}(4\tan\theta_{Z1}) - \frac{X}{L^2}(16\tan\theta_{Z1} - 8\tan\theta_{Z2}) + \frac{X^2}{L^3}(12\tan\theta_{Z1} - 12\tan\theta_{Z2}) \tag{223}$$

10.3 $\boldsymbol{N}''_X$、$\boldsymbol{N}''_Y$

$\boldsymbol{N}''_X$、$\boldsymbol{N}''_Y$ を求める。

$$N''_{X1} = \frac{4}{L^2} \tag{224}$$
$$N''_{X2} = 0 \tag{225}$$
$$N''_{X3} = 0 \tag{226}$$
$$N''_{X4} = \frac{4}{L^2} \tag{227}$$
$$N''_{X5} = 0 \tag{228}$$
$$N''_{X6} = 0 \tag{229}$$
$$N''_{X7} = -\frac{8}{L^2} \tag{230}$$

$$N''_{Y1} = \frac{1}{L^2}(12\tan\theta_{Z1} - 2\tan\theta_{Z2}) - \frac{X}{L^3}(18\tan\theta_{Z1} - 6\tan\theta_{Z2}) \tag{231}$$
$$N''_{Y2} = -\frac{6}{L^2} + \frac{X}{L^3}12 \tag{232}$$
$$N''_{Y3} = -\frac{1}{L^2}\left[4(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{1}{\cos^2\theta_{Z1}}\right] + \frac{X}{L^3}\left[6(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{1}{\cos^2\theta_{Z1}}\right] \tag{233}$$
$$N''_{Y4} = \frac{1}{L^2}(4\tan\theta_{Z1} - 6\tan\theta_{Z2}) - \frac{X}{L^3}(6\tan\theta_{Z1} - 18\tan\theta_{Z2}) \tag{234}$$
$$N''_{Y5} = \frac{6}{L^2} - \frac{X}{L^3}12 \tag{235}$$
$$N''_{Y6} = -\frac{1}{L^2}\left[2(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{1}{\cos^2\theta_{Z2}}\right] + \frac{X}{L^3}\left[6(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{1}{\cos^2\theta_{Z2}}\right] \tag{236}$$
$$N''_{Y7} = -\frac{1}{L^2}(16\tan\theta_{Z1} - 8\tan\theta_{Z2}) + \frac{X}{L^3}(24\tan\theta_{Z1} - 24\tan\theta_{Z2}) \tag{237}$$

10.4 $\dfrac{\partial \boldsymbol{N}_X}{\partial U_j}$、$\dfrac{\partial \boldsymbol{N}_Y}{\partial U_j}$

$\dfrac{\partial N_{Xi}}{\partial U_j}$ を求める。

$$\frac{\partial N_{Xi}}{\partial u_{X1}} = 0 \tag{238}$$
$$\frac{\partial N_{Xi}}{\partial u_{Y1}} = 0 \tag{239}$$
$$\frac{\partial N_{Xi}}{\partial \theta_{Z1}} = 0 \tag{240}$$
$$\frac{\partial N_{Xi}}{\partial u_{X2}} = 0 \tag{241}$$
$$\frac{\partial N_{Xi}}{\partial u_{Y2}} = 0 \tag{242}$$
$$\frac{\partial N_{Xi}}{\partial \theta_{Z2}} = 0 \tag{243}$$
$$\frac{\partial N_{Xi}}{\partial u_{X3}} = 0 \qquad (i = 1, \cdots, 7) \tag{244}$$

$\dfrac{\partial N_{Yi}}{\partial U_j}$ を求める。

$$\frac{\partial N_{Y1}}{\partial u_{X1}} = 0 \tag{245}$$
$$\frac{\partial N_{Y1}}{\partial u_{Y1}} = 0 \tag{246}$$
$$\frac{\partial N_{Y1}}{\partial \theta_{Z1}} = -\frac{X}{L}\frac{3}{\cos^2\theta_{Z1}} + \frac{X^2}{L^2}\frac{6}{\cos^2\theta_{Z1}} - \frac{X^3}{L^3}\frac{3}{\cos^2\theta_{Z1}} \tag{247}$$
$$\frac{\partial N_{Y1}}{\partial u_{X2}} = 0 \tag{248}$$
$$\frac{\partial N_{Y1}}{\partial u_{Y2}} = 0 \tag{249}$$
$$\frac{\partial N_{Y1}}{\partial \theta_{Z2}} = -\frac{X^2}{L^2}\frac{1}{\cos^2\theta_{Z2}} + \frac{X^3}{L^3}\frac{1}{\cos^2\theta_{Z2}} \tag{250}$$
$$\frac{\partial N_{Y1}}{\partial u_{X3}} = 0 \tag{251}$$

$$\frac{\partial N_{Y2}}{\partial u_{X1}} = 0 \tag{252}$$

$$\frac{\partial N_{Y2}}{\partial u_{Y1}} = 0 \tag{253}$$

$$\frac{\partial N_{Y2}}{\partial \theta_{Z1}} = 0 \tag{254}$$

$$\frac{\partial N_{Y2}}{\partial u_{X2}} = 0 \tag{255}$$

$$\frac{\partial N_{Y2}}{\partial u_{Y2}} = 0 \tag{256}$$

$$\frac{\partial N_{Y2}}{\partial \theta_{Z2}} = 0 \tag{257}$$

$$\frac{\partial N_{Y2}}{\partial u_{X3}} = 0 \tag{258}$$

$$\frac{\partial N_{Y3}}{\partial u_{X1}} = -\frac{X}{L}\frac{3}{\cos^2\theta_{Z1}} + \frac{X^2}{L^2}\frac{6}{\cos^2\theta_{Z1}} - \frac{X^3}{L^3}\frac{3}{\cos^2\theta_{Z1}} \tag{259}$$

$$\frac{\partial N_{Y3}}{\partial u_{Y1}} = 0 \tag{260}$$

$$\begin{aligned}\frac{\partial N_{Y3}}{\partial \theta_{Z1}} = \frac{X}{L}&\left[2(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}}\right] \\ &-\frac{X^2}{L^2}\left[4(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}}\right] \\ &+\frac{X^3}{L^3}\left[2(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}}\right]\end{aligned} \tag{261}$$

$$\frac{\partial N_{Y3}}{\partial u_{X2}} = -\frac{X}{L}\frac{1}{\cos^2\theta_{Z1}} + \frac{X^2}{L^2}\frac{2}{\cos^2\theta_{Z1}} - \frac{X^3}{L^3}\frac{1}{\cos^2\theta_{Z1}} \tag{262}$$

$$\frac{\partial N_{Y3}}{\partial u_{Y2}} = 0 \tag{263}$$

$$\frac{\partial N_{Y3}}{\partial \theta_{Z2}} = 0 \tag{264}$$

$$\frac{\partial N_{Y3}}{\partial u_{X3}} = \frac{X}{L}\frac{4}{\cos^2\theta_{Z1}} - \frac{X^2}{L^2}\frac{8}{\cos^2\theta_{Z1}} + \frac{X^3}{L^3}\frac{4}{\cos^2\theta_{Z1}} \tag{265}$$

$$\frac{\partial N_{Y4}}{\partial u_{X1}} = 0 \tag{266}$$

$$\frac{\partial N_{Y4}}{\partial u_{Y1}} = 0 \tag{267}$$

$$\frac{\partial N_{Y4}}{\partial \theta_{Z1}} = -\frac{X}{L}\frac{1}{\cos^2\theta_{Z1}} + \frac{X^2}{L^2}\frac{2}{\cos^2\theta_{Z1}} - \frac{X^3}{L^3}\frac{1}{\cos^2\theta_{Z1}} \tag{268}$$

$$\frac{\partial N_{Y4}}{\partial u_{X2}} = 0 \tag{269}$$

$$\frac{\partial N_{Y4}}{\partial u_{Y2}} = 0 \tag{270}$$

$$\frac{\partial N_{Y4}}{\partial \theta_{Z2}} = -\frac{X^2}{L^2}\frac{3}{\cos^2\theta_{Z2}} + \frac{X^3}{L^3}\frac{3}{\cos^2\theta_{Z2}} \tag{271}$$

$$\frac{\partial N_{Y4}}{\partial u_{X3}} = 0 \tag{272}$$

$$\frac{\partial N_{Y5}}{\partial u_{X1}} = 0 \tag{273}$$

$$\frac{\partial N_{Y5}}{\partial u_{Y1}} = 0 \tag{274}$$

$$\frac{\partial N_{Y5}}{\partial \theta_{Z1}} = 0 \tag{275}$$

$$\frac{\partial N_{Y5}}{\partial u_{X2}} = 0 \tag{276}$$

$$\frac{\partial N_{Y5}}{\partial u_{Y2}} = 0 \tag{277}$$

$$\frac{\partial N_{Y5}}{\partial \theta_{Z2}} = 0 \tag{278}$$

$$\frac{\partial N_{Y5}}{\partial u_{X3}} = 0 \tag{279}$$

$$\frac{\partial N_{Y6}}{\partial u_{X1}} = -\frac{X^2}{L^2}\frac{1}{\cos^2\theta_{Z2}} + \frac{X^3}{L^3}\frac{1}{\cos^2\theta_{Z2}} \tag{280}$$

$$\frac{\partial N_{Y6}}{\partial u_{Y1}} = 0 \tag{281}$$

$$\frac{\partial N_{Y6}}{\partial \theta_{Z1}} = 0 \tag{282}$$

$$\frac{\partial N_{Y6}}{\partial u_{X2}} = -\frac{X^2}{L^2}\frac{3}{\cos^2\theta_{Z2}} + \frac{X^3}{L^3}\frac{3}{\cos^2\theta_{Z2}} \tag{283}$$

$$\frac{\partial N_{Y6}}{\partial u_{Y2}} = 0 \tag{284}$$

$$\begin{aligned}\frac{\partial N_{Y6}}{\partial \theta_{Z2}} &= -\frac{X^2}{L^2}\left[2(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}}\right] \\ &\quad + \frac{X^3}{L^3}\left[2(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}}\right]\end{aligned} \tag{285}$$

$$\frac{\partial N_{Y6}}{\partial u_{X3}} = \frac{X^2}{L^2}\frac{4}{\cos^2\theta_{Z2}} - \frac{X^3}{L^3}\frac{4}{\cos^2\theta_{Z2}} \tag{286}$$

$$\frac{\partial N_{Y7}}{\partial u_{X1}} = 0 \tag{287}$$

$$\frac{\partial N_{Y7}}{\partial u_{Y1}} = 0 \tag{288}$$

$$\frac{\partial N_{Y7}}{\partial \theta_{Z1}} = \frac{X}{L}\frac{4}{\cos^2\theta_{Z1}} - \frac{X^2}{L^2}\frac{8}{\cos^2\theta_{Z1}} + \frac{X^3}{L^3}\frac{4}{\cos^2\theta_{Z1}} \tag{289}$$

$$\frac{\partial N_{Y7}}{\partial u_{X2}} = 0 \tag{290}$$

$$\frac{\partial N_{Y7}}{\partial u_{Y2}} = 0 \tag{291}$$

$$\frac{\partial N_{Y7}}{\partial \theta_{Z2}} = \frac{X^2}{L^2}\frac{4}{\cos^2\theta_{Z2}} - \frac{X^3}{L^3}\frac{4}{\cos^2\theta_{Z2}} \tag{292}$$

$$\frac{\partial N_{Y7}}{\partial u_{X3}} = 0 \tag{293}$$

10.5 $\frac{\partial \boldsymbol{N}'_X}{\partial U_j}$、$\frac{\partial \boldsymbol{N}'_Y}{\partial U_j}$

$\frac{\partial N'_{Xi}}{\partial U_j}$ を求める。

$$\frac{\partial N'_{Xi}}{\partial u_{X1}} = 0 \tag{294}$$

$$\frac{\partial N'_{Xi}}{\partial u_{Y1}} = 0 \tag{295}$$

$$\frac{\partial N'_{Xi}}{\partial \theta_{Z1}} = 0 \tag{296}$$

$$\frac{\partial N'_{Xi}}{\partial u_{X2}} = 0 \tag{297}$$

$$\frac{\partial N'_{Xi}}{\partial u_{Y2}} = 0 \tag{298}$$

$$\frac{\partial N'_{Xi}}{\partial \theta_{Z2}} = 0 \tag{299}$$

$$\frac{\partial N'_{Xi}}{\partial u_{X3}} = 0 \quad (i = 1, \cdots, 7) \tag{300}$$

$\dfrac{\partial N'_{Yi}}{\partial U_j}$ を求める。

$$\frac{\partial N'_{Y1}}{\partial u_{X1}} = 0 \tag{301}$$
$$\frac{\partial N'_{Y1}}{\partial u_{Y1}} = 0 \tag{302}$$
$$\frac{\partial N'_{Y1}}{\partial \theta_{Z1}} = -\frac{1}{L}\frac{3}{\cos^2\theta_{Z1}} + \frac{X}{L^2}\frac{12}{\cos^2\theta_{Z1}} - \frac{X^2}{L^3}\frac{9}{\cos^2\theta_{Z1}} \tag{303}$$
$$\frac{\partial N'_{Y1}}{\partial u_{X2}} = 0 \tag{304}$$
$$\frac{\partial N'_{Y1}}{\partial u_{Y2}} = 0 \tag{305}$$
$$\frac{\partial N'_{Y1}}{\partial \theta_{Z2}} = -\frac{X}{L^2}\frac{2}{\cos^2\theta_{Z2}} + \frac{X^2}{L^3}\frac{3}{\cos^2\theta_{Z2}} \tag{306}$$
$$\frac{\partial N'_{Y1}}{\partial u_{X3}} = 0 \tag{307}$$

$$\frac{\partial N'_{Y2}}{\partial u_{X1}} = 0 \tag{308}$$
$$\frac{\partial N'_{Y2}}{\partial u_{Y1}} = 0 \tag{309}$$
$$\frac{\partial N'_{Y2}}{\partial \theta_{Z1}} = 0 \tag{310}$$
$$\frac{\partial N'_{Y2}}{\partial u_{X2}} = 0 \tag{311}$$
$$\frac{\partial N'_{Y2}}{\partial u_{Y2}} = 0 \tag{312}$$
$$\frac{\partial N'_{Y2}}{\partial \theta_{Z2}} = 0 \tag{313}$$
$$\frac{\partial N'_{Y2}}{\partial u_{X3}} = 0 \tag{314}$$

$$\frac{\partial N'_{Y3}}{\partial u_{X1}} = -\frac{1}{L}\frac{3}{\cos^2\theta_{Z1}} + \frac{X}{L^2}\frac{12}{\cos^2\theta_{Z1}} - \frac{X^2}{L^3}\frac{9}{\cos^2\theta_{Z1}} \tag{315}$$

$$\frac{\partial N'_{Y3}}{\partial u_{Y1}} = 0 \tag{316}$$

$$\begin{aligned}\frac{\partial N'_{Y3}}{\partial \theta_{Z1}} = \frac{1}{L}&\left[2(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}}\right] \\ &-\frac{X}{L^2}\left[8(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}}\right] \\ &+\frac{X^2}{L^3}\left[6(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}}\right]\end{aligned} \tag{317}$$

$$\frac{\partial N'_{Y3}}{\partial u_{X2}} = -\frac{1}{L}\frac{1}{\cos^2\theta_{Z1}} + \frac{X}{L^2}\frac{4}{\cos^2\theta_{Z1}} - \frac{X^2}{L^3}\frac{3}{\cos^2\theta_{Z1}} \tag{318}$$

$$\frac{\partial N'_{Y3}}{\partial u_{Y2}} = 0 \tag{319}$$

$$\frac{\partial N'_{Y3}}{\partial \theta_{Z2}} = 0 \tag{320}$$

$$\frac{\partial N'_{Y3}}{\partial u_{X3}} = \frac{1}{L}\frac{4}{\cos^2\theta_{Z1}} - \frac{X}{L^2}\frac{16}{\cos^2\theta_{Z1}} + \frac{X^2}{L^3}\frac{12}{\cos^2\theta_{Z1}} \tag{321}$$

$$\frac{\partial N'_{Y4}}{\partial u_{X1}} = 0 \tag{322}$$

$$\frac{\partial N'_{Y4}}{\partial u_{Y1}} = 0 \tag{323}$$

$$\frac{\partial N'_{Y4}}{\partial u_{Z1}} = -\frac{1}{L}\frac{1}{\cos^2\theta_{Z1}} + \frac{X}{L^2}\frac{4}{\cos^2\theta_{Z1}} - \frac{X^2}{L^3}\frac{3}{\cos^2\theta_{Z1}} \tag{324}$$

$$\frac{\partial N'_{Y4}}{\partial u_{X2}} = 0 \tag{325}$$

$$\frac{\partial N'_{Y4}}{\partial u_{Y2}} = 0 \tag{326}$$

$$\frac{\partial N'_{Y4}}{\partial \theta_{Z2}} = -\frac{X}{L^2}\frac{6}{\cos^2\theta_{Z2}} + \frac{X^2}{L^3}\frac{9}{\cos^2\theta_{Z2}} \tag{327}$$

$$\frac{\partial N'_{Y4}}{\partial u_{X3}} = 0 \tag{328}$$

$$\frac{\partial N'_{Y5}}{\partial u_{X1}} = 0 \tag{329}$$
$$\frac{\partial N'_{Y5}}{\partial u_{Y1}} = 0 \tag{330}$$
$$\frac{\partial N'_{Y5}}{\partial \theta_{Z1}} = 0 \tag{331}$$
$$\frac{\partial N'_{Y5}}{\partial u_{X2}} = 0 \tag{332}$$
$$\frac{\partial N'_{Y5}}{\partial u_{Y2}} = 0 \tag{333}$$
$$\frac{\partial N'_{Y5}}{\partial \theta_{Z2}} = 0 \tag{334}$$
$$\frac{\partial N'_{Y5}}{\partial u_{X3}} = 0 \tag{335}$$

$$\frac{\partial N'_{Y6}}{\partial u_{X1}} = -\frac{X}{L^2}\frac{2}{\cos^2\theta_{Z2}} + \frac{X^2}{L^3}\frac{3}{\cos^2\theta_{Z2}} \tag{336}$$
$$\frac{\partial N'_{Y6}}{\partial u_{Y1}} = 0 \tag{337}$$
$$\frac{\partial N'_{Y6}}{\partial \theta_{Z1}} = 0 \tag{338}$$
$$\frac{\partial N'_{Y6}}{\partial u_{X2}} = -\frac{X}{L^2}\frac{6}{\cos^2\theta_{Z2}} + \frac{X^2}{L^3}\frac{9}{\cos^2\theta_{Z2}} \tag{339}$$
$$\frac{\partial N'_{Y6}}{\partial u_{Y2}} = 0 \tag{340}$$
$$\begin{aligned}\frac{\partial N'_{Y6}}{\partial \theta_{Z2}} = &-\frac{X}{L^2}\left[4(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}}\right] \\ &+\frac{X^2}{L^3}\left[6(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}}\right]\end{aligned} \tag{341}$$
$$\frac{\partial N'_{Y6}}{\partial u_{X3}} = \frac{X}{L^2}\frac{8}{\cos^2\theta_{Z2}} - \frac{X^2}{L^3}\frac{12}{\cos^2\theta_{Z2}} \tag{342}$$

$$\frac{\partial N'_{Y7}}{\partial u_{X1}} = 0 \tag{343}$$
$$\frac{\partial N'_{Y7}}{\partial u_{Y1}} = 0 \tag{344}$$
$$\frac{\partial N'_{Y7}}{\partial \theta_{Z1}} = \frac{1}{L}\frac{4}{\cos^2\theta_{Z1}} - \frac{X}{L^2}\frac{16}{\cos^2\theta_{Z1}} + \frac{X^2}{L^3}\frac{12}{\cos^2\theta_{Z1}} \tag{345}$$
$$\frac{\partial N'_{Y7}}{\partial u_{X2}} = 0 \tag{346}$$
$$\frac{\partial N'_{Y7}}{\partial u_{Y2}} = 0 \tag{347}$$
$$\frac{\partial N'_{Y7}}{\partial \theta_{Z2}} = \frac{X}{L^2}\frac{8}{\cos^2\theta_{Z2}} - \frac{X^2}{L^3}\frac{12}{\cos^2\theta_{Z2}} \tag{348}$$
$$\frac{\partial N'_{Y7}}{\partial u_{X3}} = 0 \tag{349}$$

10.6 $\dfrac{\partial \boldsymbol{N}''_X}{\partial U_j}$、$\dfrac{\partial \boldsymbol{N}''_Y}{\partial U_j}$

$\dfrac{\partial N''_{Xi}}{\partial U_j}$ を求める。

$$\frac{\partial N''_{Xi}}{\partial u_{X1}} = 0 \tag{350}$$

$$\frac{\partial N''_{Xi}}{\partial u_{Y1}} = 0 \tag{351}$$

$$\frac{\partial N''_{Xi}}{\partial \theta_{Z1}} = 0 \tag{352}$$

$$\frac{\partial N''_{Xi}}{\partial u_{X2}} = 0 \tag{353}$$

$$\frac{\partial N''_{Xi}}{\partial u_{Y2}} = 0 \tag{354}$$

$$\frac{\partial N''_{Xi}}{\partial \theta_{Z2}} = 0 \tag{355}$$

$$\frac{\partial N''_{Xi}}{\partial u_{X3}} = 0 \tag{356}$$

$\dfrac{\partial N''_{Yi}}{\partial U_j}$ を求める。

$$\frac{\partial N''_{Y1}}{\partial u_{X1}} = 0 \tag{357}$$
$$\frac{\partial N''_{Y1}}{\partial u_{Y1}} = 0 \tag{358}$$
$$\frac{\partial N''_{Y1}}{\partial \theta_{Z1}} = \frac{1}{L^2}\frac{12}{\cos^2\theta_{Z1}} - \frac{X}{L^3}\frac{18}{\cos^2\theta_{Z1}} \tag{359}$$
$$\frac{\partial N''_{Y1}}{\partial u_{X2}} = 0 \tag{360}$$
$$\frac{\partial N''_{Y1}}{\partial u_{Y2}} = 0 \tag{361}$$
$$\frac{\partial N''_{Y1}}{\partial \theta_{Z2}} = -\frac{1}{L^2}\frac{2}{\cos^2\theta_{Z2}} + \frac{X}{L^3}\frac{6}{\cos^2\theta_{Z2}} \tag{362}$$
$$\frac{\partial N''_{Y1}}{\partial u_{X3}} = 0 \tag{363}$$

$$\frac{\partial N''_{Y2}}{\partial u_{X1}} = 0 \tag{364}$$
$$\frac{\partial N''_{Y2}}{\partial u_{Y1}} = 0 \tag{365}$$
$$\frac{\partial N''_{Y2}}{\partial \theta_{Z1}} = 0 \tag{366}$$
$$\frac{\partial N''_{Y2}}{\partial u_{X2}} = 0 \tag{367}$$
$$\frac{\partial N''_{Y2}}{\partial u_{Y2}} = 0 \tag{368}$$
$$\frac{\partial N''_{Y2}}{\partial \theta_{Z2}} = 0 \tag{369}$$
$$\frac{\partial N''_{Y2}}{\partial u_{X3}} = 0 \tag{370}$$

$$\frac{\partial N''_{Y3}}{\partial u_{X1}} = \frac{1}{L^2}\frac{12}{\cos^2\theta_{Z1}} - \frac{X}{L^3}\frac{18}{\cos^2\theta_{Z1}} \tag{371}$$

$$\frac{\partial N''_{Y3}}{\partial u_{Y1}} = 0 \tag{372}$$

$$\frac{\partial N''_{Y3}}{\partial \theta_{Z1}} = -\frac{1}{L^2}\left[8(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}}\right] + \frac{X}{L^3}\left[12(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}}\right] \tag{373}$$

$$\frac{\partial N''_{Y3}}{\partial u_{X2}} = \frac{1}{L^2}\frac{8}{\cos^2\theta_{Z1}} - \frac{X}{L^3}\frac{6}{\cos^2\theta_{Z1}} \tag{374}$$

$$\frac{\partial N''_{Y3}}{\partial u_{Y2}} = 0 \tag{375}$$

$$\frac{\partial N''_{Y3}}{\partial \theta_{Z2}} = 0 \tag{376}$$

$$\frac{\partial N''_{Y3}}{\partial u_{X3}} = -\frac{1}{L^2}\frac{16}{\cos^2\theta_{Z1}} + \frac{X}{L^3}\frac{24}{\cos^2\theta_{Z1}} \tag{377}$$

$$\frac{\partial N''_{Y4}}{\partial u_{X1}} = 0 \tag{378}$$

$$\frac{\partial N''_{Y4}}{\partial u_{Y1}} = 0 \tag{379}$$

$$\frac{\partial N''_{Y4}}{\partial \theta_{Z1}} = \frac{1}{L^2}\frac{4}{\cos^2\theta_{Z1}} - \frac{X}{L^3}\frac{6}{\cos^2\theta_{Z1}} \tag{380}$$

$$\frac{\partial N''_{Y4}}{\partial u_{X2}} = 0 \tag{381}$$

$$\frac{\partial N''_{Y4}}{\partial u_{Y2}} = 0 \tag{382}$$

$$\frac{\partial N''_{Y4}}{\partial \theta_{Z2}} = -\frac{1}{L^2}\frac{6}{\cos^2\theta_{Z2}} + \frac{X}{L^3}\frac{18}{\cos^2\theta_{Z2}} \tag{383}$$

$$\frac{\partial N''_{Y4}}{\partial u_{X3}} = 0 \tag{384}$$

$$\frac{\partial N''_{Y5}}{\partial u_{X1}} = 0 \tag{385}$$

$$\frac{\partial N''_{Y5}}{\partial u_{Y1}} = 0 \tag{386}$$

$$\frac{\partial N''_{Y5}}{\partial \theta_{Z1}} = 0 \tag{387}$$

$$\frac{\partial N''_{Y5}}{\partial u_{X2}} = 0 \tag{388}$$

$$\frac{\partial N''_{Y5}}{\partial u_{X2}} = 0 \tag{389}$$

$$\frac{\partial N''_{Y5}}{\partial \theta_{Z2}} = 0 \tag{390}$$

$$\frac{\partial N''_{Y5}}{\partial u_{X3}} = 0 \tag{391}$$

$$\frac{\partial N''_{Y6}}{\partial u_{X1}} = -\frac{1}{L^2}\frac{2}{\cos^2\theta_{Z2}} + \frac{X}{L^3}\frac{6}{\cos^2\theta_{Z2}} \tag{392}$$

$$\frac{\partial N''_{Y6}}{\partial u_{Y1}} = 0 \tag{393}$$

$$\frac{\partial N''_{Y6}}{\partial \theta_{Z1}} = 0 \tag{394}$$

$$\frac{\partial N''_{Y6}}{\partial u_{X2}} = -\frac{1}{L^2}\frac{6}{\cos^2\theta_{Z2}} + \frac{X}{L^3}\frac{18}{\cos^2\theta_{Z2}} \tag{395}$$

$$\frac{\partial N''_{Y6}}{\partial u_{Y2}} = 0 \tag{396}$$

$$\begin{aligned}\frac{\partial N''_{Y6}}{\partial \theta_{Z2}} &= -\frac{1}{L^2}\left[4(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}}\right] \\ &\quad + \frac{X}{L^3}\left[12(L + u_{X1} + 3u_{X2} - 4u_{X3})\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}}\right]\end{aligned} \tag{397}$$

$$\frac{\partial N''_{Y6}}{\partial u_{X3}} = -\frac{1}{L^2}\frac{8}{\cos^2\theta_{Z2}} - \frac{X}{L^3}\frac{24}{\cos^2\theta_{Z2}} \tag{398}$$

$$\frac{\partial N''_{Y7}}{\partial u_{X1}} = 0 \tag{399}$$

$$\frac{\partial N''_{Y7}}{\partial u_{Y1}} = 0 \tag{400}$$

$$\frac{\partial N''_{Y7}}{\partial \theta_{Z1}} = -\frac{1}{L^2}\frac{16}{\cos^2\theta_{Z1}} + \frac{X}{L^3}\frac{24}{\cos^2\theta_{Z1}} \tag{401}$$

$$\frac{\partial N''_{Y7}}{\partial u_{X2}} = 0 \tag{402}$$

$$\frac{\partial N''_{Y7}}{\partial u_{Y2}} = 0 \tag{403}$$

$$\frac{\partial N''_{Y7}}{\partial \theta_{Z2}} = \frac{1}{L^2}\frac{8}{\cos^2\theta_{Z2}} - \frac{X}{L^3}\frac{24}{\cos^2\theta_{Z2}} \tag{404}$$

$$\frac{\partial N''_{Y7}}{\partial u_{X3}} = 0 \tag{405}$$

$$\tag{406}$$

10.7 $\frac{\partial^2 \boldsymbol{N}_X}{\partial U_l \partial U_m}$、$\frac{\partial^2 \boldsymbol{N}_Y}{\partial U_l \partial U_m}$

$\frac{\partial^2 \boldsymbol{N}_X}{\partial U_l \partial U_m}$ を求める。

$$\frac{\partial^2 N_{Xi}}{\partial U_l \partial U_m} = 0 \qquad (i = 0, \cdots, 7) \tag{407}$$

$\frac{\partial^2 \boldsymbol{N}_Y}{\partial U_l \partial U_m}$ を求める。非 0 項のみ示す。

$$\frac{\partial^2 N_{Y1}}{\partial {\theta_{Z1}}^2} = -\frac{X}{L}6\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X^2}{L^2}12\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X^3}{L^3}6\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{408}$$

$$\frac{\partial^2 N_{Y1}}{\partial {\theta_{Z2}}^2} = -\frac{X^2}{L^2}2\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} + \frac{X^3}{L^3}2\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{409}$$

$$\frac{\partial^2 N_{Y3}}{\partial u_{X1}\partial\theta_{Z1}} = -\frac{X}{L}6\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X^2}{L^2}12\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X^3}{L^3}6\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{410}$$

$$\frac{\partial^2 N_{Y3}}{\partial\theta_{Z1} u_{X1}} = -\frac{X}{L}6\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X^2}{L^2}12\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X^3}{L^3}6\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{411}$$

$$\begin{aligned}\frac{\partial^2 N_{Y3}}{\partial {\theta_{Z1}}^2} = &\frac{X}{L}\left[2(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{2\sin^2\theta_{Z1}+1}{\cos^4\theta_{Z1}}\right] \\ &-\frac{X^2}{L^2}\left[4(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{2\sin^2\theta_{Z1}+1}{\cos^4\theta_{Z1}}\right] \\ &+\frac{X^3}{L^3}\left[2(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{2\sin^2\theta_{Z1}+1}{\cos^4\theta_{Z1}}\right]\end{aligned} \tag{412}$$

$$\frac{\partial^2 N_{Y3}}{\partial\theta_{Z1}\partial u_{X2}} = -\frac{X}{L}2\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X^2}{L^2}4\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X^3}{L^3}2\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{413}$$

$$\frac{\partial^2 N_{Y3}}{\partial\theta_{Z1}\partial u_{X3}} = \frac{X}{L}8\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X^2}{L^2}16\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X^3}{L^3}8\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{414}$$

$$\frac{\partial^2 N_{Y3}}{\partial u_{X2}\partial\theta_{Z1}} = -\frac{X}{L}2\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X^2}{L^2}4\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X^3}{L^3}2\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{415}$$

$$\frac{\partial^2 N_{Y3}}{\partial u_{X3}\partial\theta_{Z1}} = \frac{X}{L}8\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X^2}{L^2}16\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X^3}{L^3}8\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{416}$$

$$\frac{\partial^2 N_{Y4}}{\partial {\theta_{Z1}}^2} = -\frac{X}{L}2\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X^2}{L^2}4\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X^3}{L^3}2\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{417}$$

$$\frac{\partial^2 N_{Y4}}{\partial {\theta_{Z2}}^2} = -\frac{X^2}{L^2}6\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} + \frac{X^3}{L^3}6\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{418}$$

$$\frac{\partial^2 N_{Y6}}{\partial u_{X1}\partial\theta_{Z2}} = -\frac{X^2}{L^2}2\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} + \frac{X^3}{L^3}2\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{419}$$

$$\frac{\partial^2 N_{Y6}}{\partial u_{X2}\partial\theta_{Z2}} = -\frac{X^2}{L^2}6\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} + \frac{X^3}{L^3}6\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{420}$$

$$\frac{\partial^2 N_{Y6}}{\partial\theta_{Z2}\partial u_{X1}} = -\frac{X^2}{L^2}2\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} + \frac{X^3}{L^3}2\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{421}$$

$$\frac{\partial^2 N_{Y6}}{\partial\theta_{Z2}\partial u_{X2}} = -\frac{X^2}{L^2}6\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} + \frac{X^3}{L^3}6\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{422}$$

$$\begin{aligned}\frac{\partial^2 N_{Y6}}{\partial{\theta_{Z2}}^2} &= -\frac{X^2}{L^2}\left[2(L+u_{X1}+3u_{X2}-4u_{X3})\frac{2\sin^2\theta_{Z2}+1}{\cos^4\theta_{Z2}}\right] \\ &\quad +\frac{X^3}{L^3}\left[2(L+u_{X1}+3u_{X2}-4u_{X3})\frac{2\sin^2\theta_{Z2}+1}{\cos^4\theta_{Z2}}\right]\end{aligned} \tag{423}$$

$$\frac{\partial^2 N_{Y6}}{\partial\theta_{Z2}\partial u_{X3}} = \frac{X^2}{L^2}8\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} - \frac{X^3}{L^3}8\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{424}$$

$$\frac{\partial^2 N_{Y6}}{\partial u_{X3}\partial\theta_{Z2}} = \frac{X^2}{L^2}8\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} - \frac{X^3}{L^3}8\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{425}$$

$$\frac{\partial^2 N_{Y7}}{\partial{\theta_{Z1}}^2} = \frac{X}{L}8\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X^2}{L^2}16\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X^3}{L^3}8\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{426}$$

$$\frac{\partial^2 N_{Y7}}{\partial{\theta_{Z2}}^2} = \frac{X^2}{L^2}8\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} - \frac{X^3}{L^3}8\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{427}$$

10.8 $\dfrac{\partial^2 \boldsymbol{N}'_X}{\partial U_l \partial U_m}$、$\dfrac{\partial^2 \boldsymbol{N}'_Y}{\partial U_l \partial U_m}$

$\dfrac{\partial^2 \boldsymbol{N}'_X}{\partial U_l \partial U_m}$ を求める。

$$\frac{\partial^2 N'_{Xi}}{\partial U_l \partial U_m} = 0 \quad (i = 1, \cdots, 7) \tag{428}$$

$\dfrac{\partial^2 \boldsymbol{N}'_Y}{\partial U_l \partial U_m}$ を求める。非 0 項のみ示す。

$$\frac{\partial^2 N'_{Y1}}{\partial {\theta_{Z1}}^2} = -\frac{1}{L}6\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X}{L^2}24\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X^2}{L^3}18\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{429}$$

$$\frac{\partial^2 N'_{Y1}}{\partial {\theta_{Z2}}^2} = -\frac{X}{L^2}4\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} + \frac{X^2}{L^3}6\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{430}$$

$$\frac{\partial^2 N'_{Y3}}{\partial u_{X1}\partial\theta_{Z1}} = -\frac{1}{L}6\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X}{L^2}24\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X^2}{L^3}18\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{431}$$

$$\frac{\partial^2 N'_{Y3}}{\partial\theta_{Z1}\partial u_{X1}} = -\frac{1}{L}6\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X}{L^2}24\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X^2}{L^3}18\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{432}$$

$$\begin{aligned}\frac{\partial^2 N'_{Y3}}{\partial {\theta_{Z1}}^2} &= \frac{1}{L}\left[2(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{2\sin^2\theta_{Z1}+1}{\cos^4\theta_{Z1}}\right] \\ &\quad - \frac{X}{L^2}\left[8(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{2\sin^2\theta_{Z1}+1}{\cos^4\theta_{Z1}}\right] \\ &\quad + \frac{X^2}{L^3}\left[6(L - 3u_{X1} - u_{X2} + 4u_{X3})\frac{2\sin^2\theta_{Z1}+1}{\cos^4\theta_{Z1}}\right]\end{aligned} \tag{433}$$

$$\frac{\partial^2 N'_{Y3}}{\partial\theta_{Z1}\partial u_{X2}} = -\frac{1}{L}2\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X}{L^2}8\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X^2}{L^3}6\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{434}$$

$$\frac{\partial^2 N'_{Y3}}{\partial\theta_{Z1}\partial u_{X3}} = \frac{1}{L}8\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X}{L^2}32\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X^2}{L^3}24\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{435}$$

$$\frac{\partial^2 u'_{Y3}}{\partial u_{X2}\partial\theta_{Z1}} = -\frac{1}{L}2\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X}{L^2}8\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X^2}{L^3}6\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{436}$$

$$\frac{\partial^2 N'_{Y3}}{\partial u_{X3}\partial\theta_{Z1}} = \frac{1}{L}8\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X}{L^2}32\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X^2}{L^3}24\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{437}$$

$$\frac{\partial^2 N'_{Y4}}{\partial {\theta_{Z1}}^2} = -\frac{1}{L}2\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X}{L^2}8\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X^2}{L^3}6\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{438}$$

$$\frac{\partial^2 N'_{Y4}}{\partial {\theta_{Z2}}^2} = -\frac{X}{L^2}12\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} + \frac{X^2}{L^3}18\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{439}$$

$$\frac{\partial^2 N'_{Y6}}{\partial u_{X1}\partial\theta_{Z2}} = -\frac{X}{L^2}4\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} + \frac{X^2}{L^3}6\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{440}$$

$$\frac{\partial^2 N'_{Y6}}{\partial u_{X2}\partial\theta_{Z2}} = -\frac{X}{L^2}12\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} + \frac{X^2}{L^3}18\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{441}$$

$$\frac{\partial^2 N'_{Y6}}{\partial\theta_{Z2}\partial u_{X1}} = -\frac{X}{L^2}4\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} + \frac{X^2}{L^3}6\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{442}$$

$$\frac{\partial^2 N'_{Y6}}{\partial\theta_{Z2}\partial u_{X2}} = -\frac{X}{L^2}12\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} + \frac{X^2}{L^3}18\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{443}$$

$$\begin{aligned}\frac{\partial^2 N'_{Y6}}{\partial{\theta_{Z2}}^2} &= -\frac{X}{L^2}\left[4(L+u_{X1}+3u_{X2}-4u_{X3})\frac{2\sin^2\theta_{Z2}+1}{\cos^4\theta_{Z2}}\right] \\ &+\frac{X^2}{L^3}\left[6(L+u_{X1}+3u_{X2}-4u_{X3})\frac{2\sin^2\theta_{Z2}+1}{\cos^4\theta_{Z2}}\right]\end{aligned} \tag{444}$$

$$\frac{\partial^2 N'_{Y6}}{\partial\theta_{Z2}\partial u_{X3}} = \frac{X}{L^2}16\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} - \frac{X^2}{L^3}24\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{445}$$

$$\frac{\partial^2 N'_{Y6}}{\partial u_{X3}\partial\theta_{Z2}} = \frac{X}{L^2}16\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} - \frac{X^2}{L^3}24\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{446}$$

$$\frac{\partial^2 N'_{Y7}}{\partial{\theta_{Z1}}^2} = \frac{1}{L}8\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} - \frac{X}{L^2}32\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} + \frac{X^2}{L^3}24\frac{\sin\theta_{Z1}}{\cos^3\theta_{Z1}} \tag{447}$$

$$\frac{\partial^2 N'_{Y7}}{\partial{\theta_{Z2}}^2} = \frac{X}{L^2}16\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} - \frac{X^2}{L^3}24\frac{\sin\theta_{Z2}}{\cos^3\theta_{Z2}} \tag{448}$$

11 $\boldsymbol{N}_\theta$ とその微分

11.1 $\boldsymbol{N}_\theta$

$\boldsymbol{N}'_X$、$\boldsymbol{N}'_Y$ が求まったので、(39) 式より $\boldsymbol{N}_\theta$ が求まる。

$$\begin{aligned}\boldsymbol{N}_\theta &= -\frac{u'_Y}{(1+u'_X)^2+(u'_Y)^2}\boldsymbol{N}'_X + \frac{1+u'_X}{(1+u'_X)^2+(u'_Y)^2}\boldsymbol{N}'_Y && (449)\\ &= -c_{\theta1}\boldsymbol{N}'_X + c_{\theta2}\boldsymbol{N}'_Y && (450)\\ & && (451)\end{aligned}$$

ただし、

$$c_{\theta1} = \frac{u'_Y}{(1+u'_X)^2+(u'_Y)^2} \tag{452}$$

$$c_{\theta2} = \frac{1+u'_X}{(1+u'_X)^2+(u'_Y)^2} \tag{453}$$

とおいた。

11.2 $\boldsymbol{N}'_\theta$

$$\boldsymbol{N}'_\theta = -c'_{\theta 1}\boldsymbol{N}'_X - c_{\theta 1}\boldsymbol{N}''_X + c'_{\theta 2}\boldsymbol{N}'_Y + c_{\theta 2}\boldsymbol{N}''_Y \tag{454}$$

ここで、

$$c'_{\theta 1} = \frac{-2(1+u'_X)u''_X u'_Y + (1+u'_X)^2 u''_Y - (u'_Y)^2 u''_Y}{\left((1+u'_X)^2 + (u'_Y)^2\right)^2} \tag{455}$$

$$c'_{\theta 2} = \frac{-(1+u'_X)^2 u''_X - 2u'_Y u''_Y (1+u'_X) + (u'_Y)^2 u''_X}{\left((1+u'_X)^2 + (u'_Y)^2\right)^2} \tag{456}$$

11.3 $\dfrac{\partial \boldsymbol{N}_\theta}{\partial U_j}$

$\dfrac{\partial N_{\theta i}}{\partial U_j}$ を求める。

$$\frac{\partial \boldsymbol{N}_\theta}{\partial U_j} = -\frac{\partial c_{\theta 1}}{\partial U_j}\boldsymbol{N}'_X - c_{\theta 1}\frac{\partial \boldsymbol{N}'_X}{\partial U_j} + \frac{\partial c_{\theta 2}}{\partial U_j}\boldsymbol{N}'_Y + c_{\theta 2}\frac{\partial \boldsymbol{N}'_Y}{\partial U_j} \tag{457}$$

ここで、

$$\begin{aligned}\frac{\partial c_{\theta 1}}{\partial U_j} &= -\frac{1}{((1+u'_X)^2 + (u'_Y)^2)^2}\left(2(1+u'_X)\frac{\partial u'_X}{\partial U_j} + 2u'_Y\frac{\partial u'_Y}{\partial U_j}\right)u'_Y \\ &\quad + \frac{1}{(1+u'_X)^2 + (u'_Y)^2}\frac{\partial u'_Y}{\partial U_j}\end{aligned} \tag{458}$$

$$\begin{aligned}\frac{\partial c_{\theta 2}}{\partial U_j} &= -\frac{1}{((1+u'_X)^2 + (u'_Y)^2)^2}\left(2(1+u'_X)\frac{\partial u'_X}{\partial U_j} + 2u'_Y\frac{\partial u'_Y}{\partial U_j}\right)(1+u'_X) \\ &\quad + \frac{1}{(1+u'_X)^2 + (u'_Y)^2}\frac{\partial u'_X}{\partial U_j}\end{aligned} \tag{459}$$

11.4 $\dfrac{\partial \boldsymbol{N}'_\theta}{\partial U_j}$

$\dfrac{\partial N'_{\theta i}}{\partial U_j}$ を求める。

$$\begin{aligned}\frac{\partial \boldsymbol{N}'_\theta}{\partial U_j} &= -\frac{\partial c'_{\theta 1}}{\partial U_j}\boldsymbol{N}'_X - c'_{\theta 1}\frac{\partial \boldsymbol{N}'_X}{\partial U_j} \\ &\quad -\frac{\partial c_{\theta 1}}{\partial U_j}\boldsymbol{N}''_X - c_{\theta 1}\frac{\partial \boldsymbol{N}''_X}{\partial U_j} \\ &\quad +\frac{\partial c'_{\theta 2}}{\partial U_j}\boldsymbol{N}'_Y + c'_{\theta 2}\frac{\partial \boldsymbol{N}'_Y}{\partial U_j} \\ &\quad +\frac{\partial c_{\theta 2}}{\partial U_j}\boldsymbol{N}''_Y + c_{\theta 2}\frac{\partial \boldsymbol{N}''_Y}{\partial U_j}\end{aligned} \tag{460}$$

ここで、

$$
\begin{aligned}
\frac{\partial c'_{\theta 1}}{\partial U_j} =& -2\frac{1}{((1+u'_X)^2+(u'_Y)^2)^3}\left(2(1+u'_X)\frac{\partial u'_X}{\partial U_j}+2u'_Y\frac{\partial u'_Y}{\partial U_j}\right) \\
&(-2(1+u'_X)u''_X u'_Y+(1+u'_X)^2u''_Y-(u'_Y)^2u''_Y) \\
&+\frac{1}{((1+u'_X)^2+(u'_Y)^2)^2} \\
&\left(-2\frac{\partial u'_X}{\partial U_j}u''_X u'_Y-2(1+u'_X)\frac{\partial u''_X}{\partial U_j}u'_Y-2(1+u'_X)u''_X\frac{\partial u'_Y}{\partial U_j}\right. \\
&+2(1+u'_X)\frac{\partial u'_X}{\partial U_j}u''_Y+(1+u'_X)^2\frac{\partial u''_Y}{\partial U_j} \\
&\left.-2u'_Y\frac{\partial u'_Y}{\partial U_j}u''_Y-(u'_Y)^2\frac{\partial u''_Y}{\partial U_j}\right) \qquad (461)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial c_{\theta 1}}{\partial U_j} =& -\frac{1}{((1+u'_X)^2+(u'_Y)^2)^2}\left(2(1+u'_X)\frac{\partial u'_X}{\partial U_j}+2u'_Y\frac{\partial u'_Y}{\partial U_j}\right)u'_Y \\
&+\frac{1}{(1+u'_X)^2+(u'_Y)^2}\frac{\partial u'_Y}{\partial U_j} \qquad (462)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial c'_{\theta 2}}{\partial U_j} =& -2\frac{1}{((1+u'_X)^2+(u'_Y)^2)^3}\left(2(1+u'_X)\frac{\partial u'_X}{\partial U_j}+2u'_Y\frac{\partial u'_Y}{\partial U_j}\right) \\
&(-(1+u'_X)^2u''_X-2u'_Y u''_Y(1+u'_X)+(u'_Y)^2u''_X) \\
&+\frac{1}{((1+u'_X)^2+(u'_Y)^2)^2} \\
&\left(-2(1+u'_X)\frac{\partial u'_X}{\partial U_j}u''_X-(1+u'_X)^2\frac{\partial u''_X}{\partial U_j}\right. \\
&-2\frac{\partial u'_Y}{\partial U_j}u''_Y(1+u'_X)-2u'_Y\frac{\partial u''_Y}{\partial U_j}(1+u'_X)-2u'_Y u''_Y\frac{\partial u'_X}{\partial U_j} \\
&\left.+2u'_Y\frac{\partial u'_Y}{\partial U_j}u''_X+(u'_Y)^2\frac{\partial u''_X}{\partial U_j}\right) \qquad (463)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial c_{\theta 2}}{\partial U_j} =& -\frac{1}{((1+u'_X)^2+(u'_Y)^2)^2}\left(2(1+u'_X)\frac{\partial u'_X}{\partial U_j}+2u'_Y\frac{\partial u'_Y}{\partial U_j}\right)(1+u'_X) \\
&+\frac{1}{(1+u'_X)^2+(u'_Y)^2}\frac{\partial u'_X}{\partial U_j} \qquad (464)
\end{aligned}
$$

11.5 $\dfrac{\partial^2 \boldsymbol{N}_\theta}{\partial U_l \partial U_m}$

$\dfrac{\partial^2 N_{\theta i}}{\partial U_l \partial U_m}$ を求める。

$$\begin{aligned}\frac{\partial^2 \boldsymbol{N}_\theta}{\partial U_l \partial U_m} = &-\frac{\partial^2 c_{\theta 1}}{\partial U_l \partial U_m}\boldsymbol{N}'_X - \frac{\partial c_{\theta 1}}{\partial U_l}\frac{\partial \boldsymbol{N}'_X}{\partial U_m} \\ &-\frac{\partial c_{\theta 1}}{\partial U_m}\frac{\partial \boldsymbol{N}'_X}{\partial U_l} - c_{\theta 1}\frac{\partial^2 \boldsymbol{N}'_X}{\partial U_l \partial U_m} \\ &+\frac{\partial^2 c_{\theta 2}}{\partial U_l \partial U_m}\boldsymbol{N}'_Y + \frac{\partial c_{\theta 2}}{\partial U_l}\frac{\partial \boldsymbol{N}'_Y}{\partial U_m} \\ &+\frac{\partial c_{\theta 2}}{\partial U_m}\frac{\partial \boldsymbol{N}'_Y}{\partial U_l} + c_{\theta 2}\frac{\partial^2 \boldsymbol{N}'_Y}{\partial U_l \partial U_m}\end{aligned} \tag{465}$$

ここで、

$$
\begin{aligned}
\frac{\partial c_{\theta 1}}{\partial U_m} =& -\frac{1}{((1+u'_X)^2+(u'_Y)^2)^2}\left(2(1+u'_X)\frac{\partial u'_X}{\partial U_m}+2u'_Y\frac{\partial u'_Y}{\partial U_m}\right)u'_Y \\
&+\frac{1}{(1+u'_X)^2+(u'_Y)^2}\frac{\partial u'_Y}{\partial U_m} \qquad (466)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial c_{\theta 2}}{\partial U_m} =& -\frac{1}{((1+u'_X)^2+(u'_Y)^2)^2}\left(2(1+u'_X)\frac{\partial u'_X}{\partial U_m}+2u'_Y\frac{\partial u'_Y}{\partial U_m}\right)(1+u'_X) \\
&+\frac{1}{(1+u'_X)^2+(u'_Y)^2}\frac{\partial u'_X}{\partial U_m} \qquad (467)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 c_{\theta 1}}{\partial U_l \partial U_m} =& 2\frac{1}{((1+u'_X)^2+(u'_Y)^2)^3}2((1+u'_X)^2+(u'_Y)^2)\left(2(1+u'_X)\frac{\partial u'_X}{\partial U_l}+2u'_Y\frac{\partial u'_Y}{\partial U_l}\right) \\
&\left(2(1+u'_X)\frac{\partial u'_X}{\partial U_m}+2u'_Y\frac{\partial u'_Y}{\partial U_m}\right)u'_Y \\
&-\frac{1}{((1+u'_X)^2+(u'_Y)^2)^2}\left(2\frac{\partial u'_X}{\partial U_l}\frac{\partial u'_X}{\partial U_m}+2(1+u'_X)\frac{\partial^2 u'_X}{\partial U_l \partial U_m}+2\frac{\partial u'_Y}{\partial U_l}\frac{\partial u'_Y}{\partial U_m}+2u'_Y\frac{\partial^2 u'_Y}{\partial U_l \partial U_m}\right)u'_Y \\
&-\frac{1}{((1+u'_X)^2+(u'_Y)^2)^2}\left(2(1+u'_X)\frac{\partial u'_X}{\partial U_m}+2u'_Y\frac{\partial u'_Y}{\partial U_m}\right)\frac{\partial u'_Y}{\partial U_l} \\
&-\frac{1}{((1+u'_X)^2+(u'_Y)^2)^2}\left(2(1+u'_X)\frac{\partial u'_X}{\partial U_l}+2u'_Y\frac{\partial u'_Y}{\partial U_l}\right)\frac{\partial u'_Y}{\partial U_m} \\
&+\frac{1}{(1+u'_X)^2+(u'_Y)^2}\frac{\partial^2 u'_Y}{\partial U_l \partial U_m} \qquad (468)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 c_{\theta 2}}{\partial U_l \partial U_m} =& 2\frac{1}{((1+u'_X)^2+(u'_Y)^2)^3}2((1+u'_X)^2+(u'_Y)^2)\left(2(1+u'_X)\frac{\partial u'_X}{\partial U_l}+2u'_Y\frac{\partial u'_Y}{\partial U_l}\right) \\
&\left(2(1+u'_X)\frac{\partial u'_X}{\partial U_m}+2u'_Y\frac{\partial u'_Y}{\partial U_m}\right)(1+u'_X) \\
&-\frac{1}{((1+u'_X)^2+(u'_Y)^2)^2}\left(2\frac{\partial u'_X}{\partial U_l}\frac{\partial u'_X}{\partial U_m}+2(1+u'_X)\frac{\partial^2 u'_X}{\partial U_l \partial U_m}+2\frac{\partial u'_Y}{\partial U_l}\frac{\partial u'_Y}{\partial U_m}+2u'_Y\frac{\partial^2 u'_Y}{\partial U_l \partial U_m}\right)(1+u'_X) \\
&-\frac{1}{((1+u'_X)^2+(u'_Y)^2)^2}\left(2(1+u'_X)\frac{\partial u'_X}{\partial U_m}+2u'_Y\frac{\partial u'_Y}{\partial U_m}\right)\frac{\partial u'_X}{\partial U_l} \\
&-\frac{1}{((1+u'_X)^2+(u'_Y)^2)^2}\left(2(1+u'_X)\frac{\partial u'_X}{\partial U_l}+2u'_Y\frac{\partial u'_Y}{\partial U_l}\right)\frac{\partial u'_X}{\partial U_m} \\
&+\frac{1}{(1+u'_X)^2+(u'_Y)^2}\frac{\partial^2 u'_X}{\partial U_l \partial U_m} \qquad (469)
\end{aligned}
$$

12 まとめ

幾何学的非線形を扱うために、Euler-Bernoulli 梁を Total Lagrange 法で定式化した。
変位と回転角の関係を満たす変位の補間関数 (field-consistent な補間関数) を構成した。

この補間関数は節点値に対して線形ではない。

13　参考文献

[1]P. Nanakorn, L.N. Vu, ”A 2D field-consistent beam element for large displacement analysis using the total Lagrangian formulation”, Finite Elements in Analysis and Design 42, pp.1240 -　1247, 2006